A CLOSED FORM METHOD FOR THE MEASUREMENT OF ATTENUATION IN NONLINEARLY DISPERSIVE MEDIA

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A closed form solution of the equation describing the spectrum of a Gaussian pulse propagating in a medium with nonlinear frequency dependence of attenuation is presented. This solution suggests that in general the spectrum remains Gaussian, subject only to center frequency downshift and bandwidth reduction. The possibility of experimentally determining the two nonlinear material parameters from the measurement of the center frequency downshift and the reduced bandwidth is proposed.

Key words: Bandwidth reduction; center frequency shift; frequency-dependent attenuation; Gaussian pulse; Gaussian spectrum; tissue characterization.

I. INTRODUCTION

When an ultrasound pulse passes through an attenuating medium it experiences a frequency-dependent attenuation [1-4]. The attenuation experienced at higher frequencies is generally larger than at lower frequencies. This results in changes to the spectrum after passage through a lossy medium. Specifically, the power spectrum of the pulse experiences a downshift in its center frequency [2] and a reduction in the bandwidth [5]. These changes are related to the material parameters. In order to quantitatively relate the material parameters to the observed spectral changes, it is necessary to assume models for the pulse shape and for the attenuation. Dines and Kak [6] have demonstrated the validity of modeling the ultrasonic pulse by a Gaussian function suggested by Serabian [1]. The model for the attenuation is generally assumed to be of the form $\alpha(f) = \alpha f^n$ [7] where $\alpha(f)$ is the frequency-dependent attenuation and $\alpha$ and $n$ are the material parameters. The exponent $n$ generally lies in the range $1 \leq n \leq 2$. Kuc et al. [2] have shown that for the case of $n = 1$, the pulse retains its Gaussian shape, and its bandwidth remains unchanged. For the quadratic case, Merkulova [5] has shown that, while the pulse still retains its Gaussian shape, both the bandwidth and the center frequency change.

Several ultrasonic studies on tissues [8-10] indicated that the exponent lies somewhere between 1 and 2. Ophir and Jaeger [11] considered this general case and derived an expression relating the downshifted center frequency to the material parameters. This equation [Eq. 5 of reference 11] lends itself to a numerical solution, but could not be solved in the closed form. Moreover, in this general case the shape of the pulse after propagation through the medium was not investigated. In this paper we have derived closed form expressions for the shape of the transmitted pulse and for the downshifted center frequency for this general case, by making one experimentally valid assumption. Our results show that:
1. The Gaussian spectrum retains its Gaussian form regardless of the value of \( n \).

2. In general, the bandwidth of the Gaussian pulse is reduced by its passage through the attenuating medium, in addition to a downshift in its center frequency.

3. Closed form solutions for the center frequency downshift and for the reduced bandwidth were obtained in terms of the material parameters. This suggests the possibility of determining these parameters by measuring the center frequency downshift and the reduced bandwidth of the spectrum.

II. THEORY

We assume that the incident pulse has a Gaussian shape. Its frequency spectrum is given by

\[ |I(f)| = \frac{1}{(\sqrt{2\pi})} \exp \left[-(f-f_0)^2/2\sigma^2\right], \]  

(1)

where

- \( |I(f)| \) = Fourier amplitude spectrum of the incident pulse
- \( \sigma^2 \) = the variance of the spectrum
- \( f_0 \) = the center frequency of the spectrum.

The attenuation of sound is in general nonlinearly dispersive with frequency and is of the exponential form given by

\[ |H(f)| = \exp \left[-2\alpha_o f^n Z\right], \]  

(2)

where

- \( |H(f)| \) = frequency-dependent attenuation function
- \( Z \) = total pulse propagation distance
- \( n \) = exponent of frequency dependence
- \( \alpha_o \) = amplitude attenuation coefficient of the medium.

After the pulse traverses through the medium, its amplitude spectrum is modified and is given by

\[ |R(f)| = |I(f)||H(f)| = \frac{1}{\sqrt{2 \pi}} \exp \left[\frac{-(f-f_0)^2}{2\sigma^2}\right] \exp \left[-2\alpha_o f^n Z\right]. \]  

(3)

As long as \( 0 < f < 2f_0 \), we can expand \( f^n \) as:

\[ f^n = (f_0 - f)^n = f_0^n \left[1 - \frac{f}{f_0}\right]^n \]  

(4)

Combining Eqs. (3) and (4) and performing a few algebraic manipulations, we have

\[ R(f) = \frac{1}{\sqrt{2 \pi}} \exp (d) \exp \left[\frac{1+4\alpha_o^2 \alpha Z}{2\sigma^2}\left\{f - \frac{4\alpha_o^2 \alpha Z}{1+4\alpha_o^2 \alpha Z}\right\}^2\right]. \]  

(5)
where
\[
a = \frac{n(n-1) f_o^{n-2}}{2}
\]
\[
b = n(2-n) f_o^{n-1}
\]
\[
c = \left[\frac{n^2}{2} - 2n+1\right] f_o^n
\]
\[
d = 4c \sigma_o^2 Z - \frac{(f_o - 4b \sigma_o^2 Z)^2}{1+4a \sigma_o^2 Z}. \tag{6}
\]

Eq. (5) describes a Gaussian function with its center frequency and variance given by
\[
f_c = \frac{f_o - 4b \sigma_o^2 Z}{1+4a \sigma_o^2 Z} \tag{7}
\]
\[
\sigma_c^2 = \frac{\sigma^2}{1+4a \sigma_o^2 Z}. \tag{8}
\]

Thus, as a result of the interaction of the ultrasound pulse with the medium, the center frequency and the variance are altered but the Gaussian shape is retained. Substituting for \(a\) and \(b\) into Eqs. (7) and (8), we have
\[
f_c = \frac{f_o - 2n(2-n)}{1+2n(n-1)} f_o^{n-1} \sigma_o^2 Z \tag{9}
\]
and
\[
\sigma_c^2 = \frac{\sigma^2}{1+2n(n-1)} f_o^{n-2} \sigma_o^2 Z. \tag{10}
\]

1. Linear Case \((n = 1)\)

In this case we have from Eqs. (9) and (10)
\[
f_c = f_o - 2 \sigma_o Z \sigma^2 \tag{11}
\]
\[
\sigma_c^2 = \sigma^2. \tag{12}
\]

This result is identical to what has been reported by Kuc et al. [2].

2. Quadratic Case \((n = 2)\)

In this case Eqs. (9) and (10) reduce to
\[
f_c = \frac{f_o}{1+4 \sigma_o^2 Z} \tag{13}
\]
and
\[
\sigma_c^2 = \frac{\sigma^2}{1+4 \sigma_o^2 Z}. \tag{14}
\]
These equations are identical to those reported by Merkulova [5].

Finally we show that Eq. (9) can be obtained from the expression reported by Ophir and Jaeger [11] viz.,

\[ 2n\alpha_0\gamma^2 f_c^n - f_c - f_o = 0. \]  \hspace{1cm} (15)

The frequency downshift \( f_c^n - f_o^n \) is generally small. We may therefore expand \( f_c^n - f_o^n \) in terms of \( (f_c^n - f_o^n)/f_o^n \) and ignore quadratic and higher order terms. We therefore have

\[ f_o^{n-1} = f_o^{n-1} \left[ 1 - (n-1) \frac{(f_o^n - f_c^n)}{f_o^n} \right]. \]  \hspace{1cm} (16)

Combining Eqs. (15) and (16) and rearranging terms we have

\[ f_c = \frac{f_o - 2n(2-n) f_o^{n-1} \alpha_0^2 \gamma^2}{1 + 2n(n-1) f_o^{n-2} \alpha_0^2 \gamma^2} \]  \hspace{1cm} (17)

which is identical to Eq. (9).

It is therefore clear that the approximation \( f < 2f_o \) made in deriving Eq. (9) is equivalent to the approximation in which higher order terms like \((n-1)(n-2)(f_o^n - f_c^n)^2/2f_o^2\) are neglected. It is instructive to compute the error introduced due to this assumption using typical experimental values. If we assume that \( n = 1.2 \) and \( (f_o^n - f_c^n)/f_o^n = 0.2 \), the quadratic term amounts to 0.32 percent and is thus negligible. The fact that the exact expression for the cases of \( n = 1 \) and \( 2 \) could be obtained from Eq. (9) lends support to this approximation.

III. CONCLUSIONS

In this paper we have shown the following:

1. A Gaussian pulse propagating in a lossy medium suffers a center frequency downshift and a decrease in its bandwidth except in the linear case as shown by Kuc [2].

2. The analytical form of the downshifted spectrum tends to be well approximated by a Gaussian function for \( 1 \leq n \leq 2 \).

3. We have derived closed forms for the downshifted center frequency and bandwidth, in terms of the material parameters \( \alpha_0 \), \( n \). These expressions were shown to be valid under reasonable conditions.

4. The previous result suggests that it is possible to determine the attenuation parameters of the medium by measuring the center frequency and the bandwidth of the downshifted spectrum. This technique appears to be experimentally and computationally more efficient than a previous one [11], where multiple measurements of the center frequency downshift at different transmitted bandwidths were necessary.

REFERENCES

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