SPECTRAL SHIFTS OF ULTRASONIC PROPAGATION: A STUDY OF THEORETICAL AND EXPERIMENTAL MODELS

P. A. Narayana and J. Ophir

Department of Radiology
The University of Texas Medical School
6431 Fannin St.
Houston, TX 77030

The theoretical relationship between center frequency downshift and the spectral bandwidth was investigated for pulses with a sinc(ω) spectrum propagating through lossy media. Power law and exponential models for frequency dependence of attenuation were used. Six target materials encompassing a range of attenuation parameters were used to verify the theoretical model. The frequency downshift data from these materials was used to calculate their respective attenuation parameters. It was shown theoretically and verified experimentally that for small frequency downshifts, the sinc(ω) model yields the same material parameters as the Gaussian model. The choice of the model for the attenuation of the material was found to be inconsequential.

Key words: Attenuation; attenuation models; Gaussian model; material parameters; olive oil; rectangular pulse; spectral shift; tissue equivalent material.

I. INTRODUCTION

It is well known that the attenuation of ultrasound passing through a lossy medium is frequency dependent [1-4]. The functional dependence can be modeled either by a power law of the type $\alpha = \alpha f^n$, or by an exponential of the type $\alpha = e^{\alpha' f}$, where the constants $\alpha$, $n$, and $\alpha'$, $n'$ are characteristics of the attenuation medium [5], $\alpha$ is the attenuation coefficient of the material and $f$ is the frequency. A known method for determining these parameters involves examining the spectrum of the transmitted ultrasound pulse. Since the attenuation experienced at high frequencies is generally larger than at low frequencies, a wideband ultrasonic pulse which propagates through such a medium will be distorted. This results in a center frequency downshift of its spectrum. For materials with a linear frequency dependence, this frequency shift is proportional to the integrated attenuation [2] and to the square of the bandwidth of the pulse [6]. In materials which exhibit a nonlinear frequency dependence of attenuation, it has been shown that the degree of frequency downshift is a more complex function of the characteristics of the pulse and the material [6].

A calculation of the frequency downshift must involve mathematical models for both ultrasonic pulse and the attenuating material. The models which have been used in the literature involve a Gaussian pulse spectrum and a linear or nonlinear power law frequency dependence of the attenuation in the target material [2,3,6]. Because of the nonlinear frequency dependence of the attenuation, multiple measurements must be made at numerous center frequencies or bandwidths. Thus good control of these spectral parameters must be exercised. In general, such control over the bandwidth and the center frequency is difficult to achieve for a
Gaussian spectrum. On the other hand, by exciting a wide band ultrasonic transducer with a rectangular burst obtainable from a standard signal generator, it is possible to vary both the bandwidth and the center frequency of the spectrum, while maintaining the integrity of the sinc(x) spectral shape. Typical spectra are shown in figure 1.

In this paper we have investigated the behavior of the spectrum of a rectangular pulse, in conjunction with two attenuation models. For the attenuation models we have used the power law and the exponential frequency dependences.

The attenuation values of six target phantoms were investigated. These included olive oil, human blood and four distinct samples of tissue equivalent material (TEM) which covered a range of material parameters. This was done by performing narrowband substitution measurements at several frequencies and fitting power law and exponential curves to the data [6]. Frequency shift measurements were then performed on the samples by observing the peak frequency downshift of the pulse spectrum as a function of bandwidth. The theoretical models which were appropriate for both combinations of the pulse and material formulations were fitted to the experimental data points. We have found that for all targets investigated, the frequency shift measurements interpreted using these models yielded material parameters which were in good agreement with those found from substitution measurements. We have also shown that under certain conditions, this frequency spectrum could be approximated by a Gaussian spectrum. In addition, we have observed that both power law and exponential models gave comparably consistent results.

Fig. 1 Spectra obtained by exciting a wideband transducer with a rectangular burst of sinewaves. The four spectra correspond to different numbers of cycles contained in the rectangular burst. a) 4, b) 5, c) 6 and d) 8 cycles. The center frequency used was 4.5 MHz. Horizontal scale is 500 KHz/div; vertical scale is 10 dB/div. The frequency markers superimposed on the spectra, are located 1 MHz apart. Note that the integrity of the sinc(x) shape is well maintained.
II. THEORY

Let us consider a rectangular ultrasonic pulse with an amplitude $A$, width $T_0$, and center frequency $f_0$. The frequency spectrum of such a pulse is given by [7]

$$T(f) = \frac{A^2 T_0 \sin\pi T_0 (f-f_0)}{\pi T_0 (f-f_0)}.$$  \hspace{1cm} (1)

On the other hand if we assume a Gaussian pulse with variance $\sigma^2$, its frequency spectrum is given by

$$T(f) = \left(\frac{1}{\sigma \sqrt{2\pi}}\right) e^{-\frac{(f-f_0)^2}{2\sigma^2}}.$$ \hspace{1cm} (2)

It is assumed that the attenuation of sound in the medium of propagation is nonlinearly dispersive with frequency, and is of the exponential form

$$H(f) = e^{-\alpha(f)Z},$$  \hspace{1cm} (3)

where $\alpha(f)$ is the attenuation coefficient and $Z$ is the total pulse propagation distance. $\alpha$ is a function of frequency and can be expressed as

$$\alpha = \alpha_o f^n, \text{ or } \alpha = \alpha_o' e^{nf}.$$  

In a lossy medium, the received spectrum is therefore given by

$$R(f) = T(f) H(f).$$ \hspace{1cm} (4)

After traversing a path in the medium, the pulse has experienced a downshift in its center frequency. In order to find the downshifted center frequency we differentiate $R(f)$ with respect to $f$ and equate the result to zero. In the following we present the necessary expressions for various models.

1. Gaussian Pulse

a. Power law model

This has been discussed by Ophir and Jaeger [6]. The downshifted center frequency $f_c$ is given by the relationship

$$2n\alpha_o \sigma^2 f_c^{n-1} + f_c - f_0 = 0.$$ \hspace{1cm} (5)

b. Exponential model

The frequency dependence of attenuation is assumed to be represented by

$$\alpha = \alpha_o' e^{nf}.$$  \hspace{1cm} (6)

The downshifted center frequency is given by (See appendix)

$$\alpha_o' \sigma n' f_c e^{n'f_c \sigma^2} + f_c - f_0 = 0.$$  \hspace{1cm} (7)
2. Rectangular Pulse

a. Power law model:

The spectrum is given by Eq. (2).

The downshifted center frequency is given by the relation (See appendix)

\[
\cos^2 T_o (f_c - f_o) - \frac{\sin^2 T_o (f_c - f_o)}{\pi T_o (f_c - f_o)} \left[ n Z f_c^{n-1} (f_c - f_o) + 1 \right] = 0, \tag{8}
\]

In the special case of \( \pi T_o (f_c - f_o) \ll 1 \), corresponding to small frequency downshifts due to small values of the material parameters, we have

\[
\frac{\sin T_o (f_c - f_o)}{\pi T_o (f_c - f_o)} \sim 1 \tag{9}
\]

and

\[
\cos^2 T_o (f_c - f_o) \sim 1 - \frac{1}{2} \left[ \pi T_o (f_c - f_o) \right]^2. \tag{10}
\]

Eq. (8) can be rewritten as

\[
\frac{2 n Z f_c^{n-1}}{(\pi T_o)^2} + (f_c - f_o) = 0. \tag{11}
\]

If we identify \( \frac{\sqrt{2}}{\pi T_o} = \sigma \), Eq. (11) can be rewritten as

\[
2 n Z \sigma^2 f_c^{n-1} + (f_c - f_o) = 0. \tag{12}
\]

This is identical to Eq. (5). Thus in the special case of \( \pi T_o (f_c - f_o) \ll 1 \), the spectrum can be approximated by a Gaussian.

b. Exponential model:

The downshifted center frequency is given by the relation

\[
\cos^2 T_o (f_c - f_o) - \frac{\sin^2 T_o (f_c - f_o)}{\pi T_o (f_c - f_o)} \left[ \alpha_n f_c^{n-1} (f_c - f_o) + 1 \right] = 0, \tag{13}
\]

III. MATERIALS AND METHODS

Two experimental procedures have been used to verify the theory.

The first procedure involved the determination of the frequency dependent attenuation of the material using a narrowband pulse echo substitution method in the range of 2 to 6 MHz. The second procedure involved the measurement of the center frequency downshift from the spectrum analyzer output for various pulse widths. The center frequency employed was 4.5 MHz and the pulse width was varied from \( \approx 0.2 \) to 3 \( \mu s \). This range corresponds
to sinewave bursts of 1 and 12 cycles. The values of \((\alpha, n)\) and \((\alpha', n')\) obtained from the second experiment were compared with the corresponding values determined from the first experiment. The experimental setup was described elsewhere [6].

Six target materials have been investigated. These were olive oil\(^1\), blood (outdated blood from the blood bank) and four distinct tissue equivalent materials.\(^2\) These materials exhibit a reasonably wide range of material parameters. The sample holders were made of plexiglas in the form of cylinders. The cylinder thickness was typically 5 cm. The sample holders were closed on both sides with 0.5 mil thick mylar membranes. All samples were degassed to remove the trapped air. The sample holder was placed in the water tank with a 1/2" thick plexiglas bottom. The bottom of the tank was in the focal region of the transducer, at a range of 12 cm. The echo from the front surface of the bottom was monitored. The energy loss due to impedance mismatch between the target material and water was found to be small and was ignored. The pulse width \(T_0\) was determined from the recorded spectrum in water, assumed to be nonattenuating.

IV. RESULTS

The values of \((\alpha, n)\) and \((\alpha', n')\) determined for six target materials using narrowband pulse echo substitution method are shown in Table 1. The values of \(\alpha\) and \(n\) were obtained by fitting the observed attenuation at various frequencies to a power law using a least squares routine. The values of \(\alpha'\) and \(n'\) were obtained by fitting the experimental values to an exponential curve. The errors given in this table correspond to the range of values obtained from a number of measurements.

The spectral shift was measured for all target materials at several pulsewidths \(T_0\). The appropriate equation which relates the material parameters to the frequency downshift and to the pulsewidth (i.e. (5), (7), (12) or (13)) was fitted to the data points for each target material. The material parameters were obtained from this fit. These values are shown in Table 1. The estimated error in the determination of the material parameters obtained using this method is 10 percent.

IV. DISCUSSION

The results suggest that it is possible to determine the values of \((\alpha_0, n)\) and \((\alpha', n')\) from the spectral shift data. Comparison of the experimental values for these parameters (Table 1) indicates their insensitivity to the choice of attenuation model.

It is generally easy to generate and apply an accurate rectangular excitation to the transducer. It is thus logical to analyze the spectral shifts using this model. We have shown that this model can predict the material parameters from the spectral shift data. The added feature of this model is that the usable bandwidth is large if sidelobes are considered. This model could easily be extended to the calculation of the frequency downshift of the sidelobes. For a given target material, greater frequency downshift can be expected for higher order sidelobes.

We have shown (Eqs.8-12) that under certain conditions, the spectrum of a rectangular pulse could be approximated by a Gaussian. This was ver-

\(^1\)Arrowhead Mills, Hereford, TX

\(^2\)Acoustic Standards Corp., Houston, TX
Table 1. The values of material parameters determined from the substitution method, and from spectral shift data. Power and exponential law models of attenuation are shown.

### POWER LAW MODEL

<table>
<thead>
<tr>
<th>SAMPLE</th>
<th>( a_o^{**} )</th>
<th>( n )</th>
<th>( r^2^{*} )</th>
<th>( a_o^{**} )</th>
<th>( n )</th>
<th>( a_o^{**} )</th>
<th>( n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Olive oil</td>
<td>0.011 ± 0.002</td>
<td>1.89 ± 0.05</td>
<td>0.999</td>
<td>0.03 ± 1.33</td>
<td>0.014</td>
<td>1.86</td>
<td></td>
</tr>
<tr>
<td>Blood</td>
<td>0.028 ± 0.002</td>
<td>1.19 ± 0.02</td>
<td>0.997</td>
<td>0.012 ± 1.04</td>
<td>0.028</td>
<td>1.36</td>
<td></td>
</tr>
<tr>
<td>TEM</td>
<td>0.028 ± 0.004</td>
<td>1.38 ± 0.05</td>
<td>0.991</td>
<td>0.022 ± 1.49</td>
<td>0.031</td>
<td>1.32</td>
<td></td>
</tr>
<tr>
<td>TEM2</td>
<td>0.04 ± 0.003</td>
<td>1.52 ± 0.05</td>
<td>0.989</td>
<td>0.025 ± 1.68</td>
<td>0.027</td>
<td>1.56</td>
<td></td>
</tr>
<tr>
<td>TEM3</td>
<td>0.05 ± 0.005</td>
<td>1.32 ± 0.05</td>
<td>0.968</td>
<td>0.03 ± 1.20</td>
<td>0.04</td>
<td>1.28</td>
<td></td>
</tr>
<tr>
<td>TEM4</td>
<td>0.05 ± 0.004</td>
<td>1.22 ± 0.04</td>
<td>0.971</td>
<td>0.06 ± 1.10</td>
<td>0.05</td>
<td>1.28</td>
<td></td>
</tr>
</tbody>
</table>

### EXPONENTIAL LAW MODEL

<table>
<thead>
<tr>
<th>SAMPLE</th>
<th>( a_o^{**} )</th>
<th>( n'^{++} )</th>
<th>( r^2^{*} )</th>
<th>( a_o^{**} )</th>
<th>( n' )</th>
<th>( a_o^{**} )</th>
<th>( n'^{++} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Olive oil</td>
<td>0.019 ± 0.003</td>
<td>0.47 ± 0.01</td>
<td>0.980</td>
<td>0.015 ± 0.37</td>
<td>0.018</td>
<td>0.45</td>
<td></td>
</tr>
<tr>
<td>Blood</td>
<td>0.045 ± 0.005</td>
<td>0.27 ± 0.01</td>
<td>0.988</td>
<td>0.052 ± 0.14</td>
<td>0.053</td>
<td>0.22</td>
<td></td>
</tr>
<tr>
<td>TEM</td>
<td>0.05 ± 0.005</td>
<td>0.38 ± 0.03</td>
<td>0.988</td>
<td>0.023 ± 0.41</td>
<td>0.049</td>
<td>0.40</td>
<td></td>
</tr>
<tr>
<td>TEM2</td>
<td>0.043 ± 0.002</td>
<td>0.47 ± 0.01</td>
<td>0.995</td>
<td>0.021 ± 0.42</td>
<td>0.049</td>
<td>0.41</td>
<td></td>
</tr>
<tr>
<td>TEM3</td>
<td>0.05 ± 0.006</td>
<td>0.38 ± 0.02</td>
<td>0.996</td>
<td>0.025 ± 0.42</td>
<td>0.051</td>
<td>0.39</td>
<td></td>
</tr>
<tr>
<td>TEM4</td>
<td>0.06 ± 0.004</td>
<td>0.35 ± 0.02</td>
<td>0.993</td>
<td>0.031 ± 0.32</td>
<td>0.055</td>
<td>0.36</td>
<td></td>
</tr>
</tbody>
</table>

* \( r^2 \) is the coefficient of determination. \( r^2 \approx 1 \) indicates a good fit.
** the units of \( a_o \) are Nepers cm\(^{-1}\) MHz\(^{-n}\)
† the units of \( a_o' \) are Nepers cm\(^{-1}\)
++ the units of \( n' \) are MHz\(^{-1}\)

Ified experimentally (table 1). In the case of olive oil, however, it is seen that the error in this approximation becomes large. This is an expected result due to the breakdown of the condition \( | \mathcal{N}_T (f-f_0) | < 1 \) for materials which exhibit a large degree of spectral downshift due to large attenuation.
APPENDIX

In this appendix we derive Eq. (8) in detail. The other equations can be derived in a similar manner.

We make two assumptions, viz a) the frequency dependence of attenuation obeys a power law i.e., \( \alpha = \alpha_o f^n \) and b) the shape of the incident pulse is rectangular.

The frequency spectrum of a rectangular pulse is given by

\[
T(f) = \frac{\sin \pi T_o (f - f_o)}{\pi T_o (f - f_o)}.
\]  \hspace{1cm} (A1)

If we make the substitution \( a = \pi T_o \) and \( \theta = f - f_o \) Eq. (A1) can be rewritten as

\[
T(f) = \frac{\sin (a \theta)}{a \theta} = \frac{1}{2ia \theta} \left[ e^{ia \theta} - e^{-ia \theta} \right].
\]  \hspace{1cm} (A2)

The attenuation by the medium is described by

\[
H(f) = e^{-\alpha Z} = e^{-\alpha_o Z f^n},
\]  \hspace{1cm} (A3)

where \( Z \) is the total pulse propagation distance expressed in cm and the frequency \( f \) is expressed in MHz.

In an attenuating medium, the received spectrum \( R(f) \) is given by

\[
R(f) = T(f) H(f).
\]  \hspace{1cm} (A4)

Substituting for \( T(f) \) and \( H(f) \) from (A2) and (A3) respectively, we have

\[
R(f) = \frac{1}{2ia \theta} \left[ e^{ia \theta} - e^{-ia \theta} \right] e^{-\alpha_o Z f^n}
\]  \hspace{1cm} (A5)

The center frequency \( f_c \) is defined as the frequency where the slope of the spectrum equals 0. \( f_c \) is obtained by differentiating Eq. (A5) with respect to \( f \) and equating it to 0.

\[
\frac{dR(f)}{df} = \frac{1}{\theta} (ia - \nu_o Z f_c n^{-1}) e^{ia \theta} - (-ia - \nu_o Z f_c n^{-1}) e^{-ia \theta}
\]

\[
- \frac{1}{\theta} e^{ia \theta} (e^{ia \theta} - e^{-ia \theta}) = 0
\]  \hspace{1cm} (A6)

Collecting terms, we get

\[
\frac{1}{\theta} \left[ ia ( e^{ia \theta} + e^{-ia \theta} ) - \nu_o Z f_c n^{-1} ( e^{ia \theta} - e^{-ia \theta} ) \right] - \frac{1}{\theta ^2} ( e^{ia \theta} - e^{-ia \theta} ) = 0
\]  \hspace{1cm} (A7)

or, equivalently

\[
\frac{1}{\theta} \left[ i a \cos (a \theta) - \nu_o Z f_c n^{-1} \sin (a \theta) \right] - \frac{i}{\theta ^2} \sin (a \theta) = 0.
\]  \hspace{1cm} (A8)
Substituting for \( a \) and \( \theta \) we have

\[
\cos \pi T_o \left( f_c - f_o \right) = \frac{\sin \pi T_o \left( f_c - f_o \right)}{\pi T_o \left( f_c - f_o \right)} \left[ \frac{\ln \left( \frac{f_c}{f_o} \right)^{n-1}}{n} \right] = 0. \tag{A9}
\]

This is the expression given by Eq. (8).

REFERENCES


