Estimation of the Speed of Ultrasound Propagation in Biological Tissues: A Beam-Tracking Method

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Abstract—A pulse-echo beam-tracking method for the estimation of the speed of sound in tissue is described. We have shown that precision on the order of ±0.5 percent and accuracy on the order of 1 percent are obtainable in uniformly scattering foam phantoms using several echo-waveforms which are several centimeters long. The precision could be improved if 1) more uniform tissues or phantoms are used, 2) more uncorrelated echo-waveforms are examined, and 3) the length of each echo-waveform is extended. The potential effects due to refractive layers on the estimation are discussed.

I. INTRODUCTION

THE SPEED of ultrasound propagation in soft tissues is thought to have a mean value of about 1540 m s⁻¹ [1]. Moreover, the variation in the speed of ultrasound around this mean value is rather small, on the order of ±5 percent. The speed of sound in fat is at the low end of the spectrum (≈1460 m s⁻¹) [2]–[4], and thus in fat rich tissue, such as postmenopausal breast, it tends to be low as well (≈1440–1490 m s⁻¹). The speed of sound in premenopausal breast, on the other hand, has a range of ≈1490–1550 m s⁻¹ [4]. At the high end of the spectrum, the speed of sound in muscle has been reported to be in the range of ≈1550–1640 m s⁻¹ [2]–[5]. Data for pathological tissue are quite rare in the literature, with some noted exceptions [6], [7]. It must be stressed that the overwhelming bulk of the speed-of-sound data reported in the literature are derived from excised tissue in vitro, and as such some questions linger as to the strict validity of these data when applied to the in vivo case.

In principle, the speed of sound in tissue can be measured using either pulse-echo or transmission methods. Unfortunately, these measurements are difficult even when recourse is made to transmission methods [8].

Transmission methods of various kinds [1] have traditionally been used by many investigators in laboratory experiments. These include pulse transit time measurements [9], the sing-around technique [10], and the velocity difference method [11]. Transmission methods are adaptable to in vivo cases only if good access is possible from many angles and when no strongly refracting structures exist. Moreover, simple transmission techniques generally measure the transit time of the primary sound pulse as it passes through several different tissues, which could have different speeds of sound, and the measurement of this quantity inside a particular tissue of interest cannot be made. A notable exception of this limitation is time-of-flight tomography [12], whereby a two-dimensional speed of sound map is produced from projection data. It is, however, necessary to have a wide angle acoustic access and a large investment in on-line data processing equipment.

Pulse-echo techniques using reference reflectors [1] suffer from essentially the same limitations as the transmission techniques. However, Robinson et al. [13] have made use of the fact that if an identifiable target within the tissue is scanned from two directions, the resultant images of this target would adequately overlap only if the speed of sound propagation which is assumed by the scanner matches its actual value; the greater the mismatch, the less satisfactory the overlap will be. Cross-correlation techniques are used on the images in order to determine the actual deviation from perfect overlap, and the actual speed of sound in the tissue is computed with repeatability on the order of one percent. This technique has demonstrated the feasibility of obtaining speed of sound measurement in vivo using pulse-echo techniques; however, it suffers from the need to locate a reliable target in the tissue, the effects of refraction, the inability to localize the measurement, and the need for elaborate processing. Very recently, several novel techniques have been described for the measurement of sound speed in vivo using pulse-echo techniques. Bamber [14] has described an array scanner that, in conjunction with a biprism, generates two divergent images of the same target. As in Robinson’s technique, the separation between the images of the target are an indication of the global speed of sound between the transducer face and the target. Bamber reports current error in accuracy on the order of eight percent, which is inadequate for tissue differentiation.

Another technique that also relies on clarifying image distortion has been reported by Katamura et al. [15] and by Hayashi et al. [16]. Both authors use the delay lines associated with focused arrays to adjust for the sharpest image. From the delays so obtained, the speed of sound between the transducer and the target is determined. Katamura [15] used the technique successively at various depths and obtains local differential measurements inside the tissue, which could be prone to large errors. Studies in the liver by Hayashi show a ±2 percent repeatability in vivo, while the repeatability of the differential technique is not reported.

An alternative technique for measuring the speed of sound...
sound in tissue between two transducers is the crossed beam method [17], [18]. Simply stated, two transducers are arranged so that their axes of radiation intersect. The time of flight of the transmitted pulse to the intersecting volume plus the time of flight of the backscattered echoes returning to the receiving transducer is computed. Assuming negligible effect of the speed-of-sound propagation in the body wall fat layer (which could be large if proximal regions are investigated), the total path length (assumed from geometric considerations) is divided by the total travel time to yield the speed of sound in tissue.

Ohitsuki et al. [19] have reported a technique they call the reference point technique, which uses two reference points such as the two edges of a tumor. The actual distance between the points is measured along one axis using a linearly scanned image, while the time of flight in the tissue contained between the reference points is measured with an A-mode transducer at right angles. They report an accuracy error in the measurement of about ten percent.

In this paper we report a pulse-echo estimation technique, whereby the speed of sound along an arbitrary line (or in an arbitrary area or volume) in tissue can be found. Some of the experimental evidence in phantoms indicates that a repeatability of 1/2 percent and an accuracy of about one percent are possible utilizing only a small number of echo waveforms that are several centimeters long.

II. Beam-Tracking Method

The basic method is illustrated on Fig. 1. Two coplanar transducers are positioned such that their beam axes intersect at right angles. One of the transducers serves as a transmitter, while the other as a receiver. A short pulse is first emitted from the transmitting transducer. Energy scattered from the volume of beam intersection $V_1$ arrives at the receiving transducer located at position $R_1$ at time $t_1$ after the emission of the pulse. The receiving transducer is then moved to a new location $R_2$, separated from the previous location $R_1$ by a distance $\Delta x$. The new arrival time of the scattered energy from volume $V_2$ is now $t_2$, where $\Delta t_1 = t_2 - t_1$. The receiving transducer is moved again to location $R_3$, and the quantity $\Delta t_2 = t_3 - t_1$ is again measured, and so on. The tracking interval $\Delta x$, which is under user control, depends on the characteristics of the transducer radiation field and the scattering properties of the target; a typical value of $\Delta x = 1$ mm. After the receiving transducer has been positioned in $n$ locations and $n - 1$ values of $\Delta t$ are acquired (where typically $10 \leq n - 1 \leq 100$), a plot is made of $\Delta t$ vs. $\Delta x$.

A least squares linear regression fit is performed, and the slope of this fit is $\hat{c}$, where $\hat{c}$ is the estimate of the speed of sound propagation along the path of the transmitted pulse. Variations in the speed of sound along this path are averaged out by the linear regression process.

It is clear that many modifications can be made to this basic concept. Some of the important ones are as follows:

1) The roles of the transmitter and receiver can be interchanged; that is, the transmitting transducer is moved, and the receiving transducer is stationary.

2) Either one or both transducers can be replaced by transducer arrays. This allows a) estimating the speed of sound propagation in a volume or an area, rather than only along a line; b) rapid imaging of the area under investigation, where at least one array also operates in the transmit-receive mode; and c) elimination of the physical movement of the transducer as described above by sequencing the elements of one or both arrays. This, of course, speeds up the examination time to where the only limit on the attainable examination time becomes the range of the target and the speed of sound propagation in the overlying tissue layers.

3) The angle of intersection between the beams (Fig. 1) need not be restricted to $\pi/2$, but can assume other known values as well. This angle could also be variable throughout the scan, such as when using a phased array or a mechanically wobbling transducer.

Another possible configuration, incorporating the generalizations just described, is shown in Fig. 2. Here we have a single crystal transmitter and a receiving array where the elements are inclined (or the beam is otherwise angled) at an angle $\theta$. The time required for the pulse to travel from the transmitter to point $a$ is $t_a$. The time required for the pulse to travel from points $b$ or $d$ to the corresponding receivers $R_1$ and $R_2$ is assumed to be a constant $t_b$. (A discussion of potential errors is given in the Appendix.) Thus the total flight time for the transmitted pulse from the transmitter to point $a$ and then to receiver $R_1$ is

$$t_{R_1} = t_a + t_b + \Delta y/\hat{c} = t_a + t_b + (\Delta x \tan \theta)/\hat{c}.$$
Similarly, the transit time of the pulse from the transmitter to point \( d \) and then to receiver \( R_2 \) is

\[
t_{R_2} = t_a + t_b + \Delta z / \hat{c} = t_a + t_b + \Delta x / (\hat{c} \cos \theta)
\]

and the difference in arrival times \( \Delta t = t_{R_1} - t_{R_1} \), is

\[
\Delta t = t_a + t_b + \frac{\Delta x}{\hat{c} \cos \theta} - t_a - t_b - \frac{\Delta x \tan \theta}{\hat{c}}
\]

\[
\hat{c} = \frac{\Delta x}{\Delta t} \left( \frac{1 - \sin \theta}{\cos \theta} \right)
\]

from which the estimated speed of sound in the medium is derived as

\[
\hat{c} = \frac{\Delta x}{\Delta t} \left( \frac{1 - \sin \theta}{\cos \theta} \right)
\]  

III. EXPERIMENT

Experimental confirmation was obtained using the apparatus shown in Fig. 4. A test phantom was constructed (details given later) and placed in a 60-gal temperature-controlled water bath. Two ultrasonic transducers (3.5 MHz, 13 mm, focused at 4-10 cm) are mounted on an x-y precision positioning system such that their axes of radiation intersected in a plane at right angles. The positioning devices are driven by stepper motors, which in turn are controlled by a custom digital controller. The uncertainty in positioning is on the order of \( \pm 10 \mu m \). In a typical experiment the transmitting transducer position is held fixed, while the receiving transducer position is incremented in 1-mm steps.

The transmitting transducer is driven by a Metrotek high-voltage pulser at a rate of 1-2 kHz. The received echoes are amplified via an input protected custom preamplifier, pass through a wideband step attenuator, and then pass through a custom RF amplifier and full wave demodulator and low-pass single-pole RC filter, which strip off the carrier and produce the echo envelope waveform. The pole frequency of the filter is adjusted to approximately 200 kHz. The output of the filter is fed to a Tektronix 7465 oscilloscope, which is triggered by the transmitter.

The received signal resembles a noisy double-sided ex-
ponentially modulated carrier waveform, with a relatively long duration on the order of 50 µs (see Fig. 5(a)). This raw RF is unsuitable for direct time delay measurements because of its spiky appearance. Upon full wave demodulation and low-pass filtering, the envelope of the signal is produced, which resembles a unipolar double sided exponential with a well defined maximum. Spurious spikes present in the raw RF waveform are eliminated (see Fig. 5(b)).

In a typical experiment, the receiving transducer is positioned at some arbitrary initial position, and the peak echo is positioned horizontally at the center of the oscilloscope display via the digital delayed-sweep adjustment control, while the attendant delay time is recorded to within 100 ns. The receiving transducer is then repositioned 1 mm away (toward or away from the transmitting transducer), and the process is repeated. Typically, 50–80 point pairs are collected. The data are then manually entered into the HP-85 computer for a least-squares linear regression analysis. Both the slope of the linear fit (in µs/mm) and the coefficient of determination \( r^2 \), where \( r^2 \) is the square of the correlation coefficient \( r \) [22, p. 341], are recorded. The value of \( r^2 = 1 \) indicates a perfect fit.

A 15.3-cm-tall 7.16-cm-diameter acrylic cylinder with 0.64-cm walls and one end closed served as the phantom container. A block of open cell reticulated polyester foam (Scott Paper Co., Chester, PA) was cut into a cylinder, which fit tightly in the container. One of two grades of foam were used: 1) foam with an average 20 pores per linear inch and 2) foam with an average 30 pores per linear inch. Both foams contain open spaces in about 98 percent of their volume. A test liquid consisting of 50 percent ACS grade glycerol by volume in distilled water was added to the container until it covered the foam block. The container was then placed in a desiccator, and laboratory vacuum was applied for \( \frac{1}{2} \) h. The container was then removed and placed in the water tank, such that its open top was slightly above the water level (Fig. 4). The aperture of the transmitting transducer was immersed in the glycerol solution. Experimental verification of the speed of ultrasound propagation in the solution was performed by using a velocimeter as described by McWhirt [20]. Briefly, the length of a glass tube containing distilled water, with a transducer at one end and a stainless steel plate at the other, was calibrated by measuring the transit time of an ultrasonic pulse passing from one end to the other. The effective length of the tube was computed from tabulated values of the speed of sound in distilled water at the proper temperatures [10]. The tube was then drained and refilled with the glycerol solution. The new transit time was measured, from which the speed of sound was computed. The transit time of the pulse was measured as the time interval between the leading edge of the transmitter trigger pulse and the time of the second positive-to-negative zero crossing of the echo from the steel plate. Since in both water and glycerol cases the measurement was done this way, the differential nature of this technique tended to cancel out any bias in the measurement.

One in vitro experiment was performed on a porcine kidney to further corroborate the beam-tracking method. The specimen was removed and stored in normal saline at 4°C for 24 h. Within 2 h prior to the experiment, the kidney and saline were transferred into a narrow upright tube constructed of 12.5-µm-thick polycarbonate film. The tube was placed in a glass desiccator and laboratory vacuum was applied for \( \frac{1}{2} \) h. The tube was then placed in the water tank and three beam-tracking estimations were performed along the long axis of the kidney, each time rotating the kidney by approximately 30°.

After the completion of these experiments, the kidney was removed from the tube and placed horizontally on an acrylic supporting pad, while a transducer connected in the pulse–echo mode was used to measure the change in the time of arrival of the back-wall echo due to the introduction of the kidney \( \Delta t \). From the knowledge of the average thickness of the kidney \( (\Delta x = 27 \pm 1 \text{ mm}) \) and the tabulated speed of sound in water at 21°C (\( = 1485.7 \text{ ms}^{-1} \)) the speed of sound \( c_{\text{tissue}} \) across the kidney was computed as [21]

\[
\frac{1}{c_{\text{tissue}}} = \frac{1}{1485.7} - \frac{\Delta t}{2\Delta x}.
\]
This value was then compared to the one obtained from the beam tracking method. The ±1-mm uncertainty in the thickness of the kidney was determined by making several caliper measurements across the kidney area located immediately in front of the transducer.

IV. Results

Nine experiments were performed on the phantom. Five of these involved the use of the 20 ppi foam, while four involved 30 ppi foam. Tables I and II summarize the results from these two groups of experiments. Figs. 6 and 7 show actual data. Fig. 6 shows data from 50-percent glycerol in 20 ppi foam, for which the worst coefficient of determination was found (experiment 1, $r^2 = 0.992$). Fig. 7 shows data from 50-percent glycerol in 30-ppi foam, for which the best coefficient of determination was found (experiment 9, $r^2 = 0.997$). The value found from the velocimeter measurements was 1793.2 ms$^{-1}$.

Table III summarizes the results of the beam-tracking experiments in porcine kidney in vitro. The average speed of sound through a 54-mm path in the kidney was estimated at $1496.6 \pm 75.9$ ms$^{-1}$ (mean ± standard deviation), while the substitution method yielded $1527.9 \pm 56.5$ ms$^{-1}$. Actual data from experiment 12 (worst $r^2 = 0.982$) is shown in Fig. 8.

V. Discussion

Many factors are involved in the performance of the method. A discussion of the more important ones is given below.

The precision of the estimation, which we define as the standard deviation of the estimator, clearly depends on such statistical parameters as the number of experiments,
the degree of correlation between them, the length of the total tracking intervals, the statistical properties of the target and the signal processing algorithm utilized.

It can be shown [22, p. 269] that in each experiment the variance of the regression slope (which is the inverse of the speed of sound estimator)

$$\text{var} \left( \frac{1}{c} \right) = \frac{\sigma^2}{\sum x_i^2}$$

(3)

where $\sigma^2$ is the variance of the residuals (the vertical excursions of the data about the regression line, assumed to be constant along the tracking interval), and the $x_i$ are the distances along the abscissa of the data points from the center of the regression line. If $m$ experiments are performed using decorrelated beams, the variance of the estimator is further reduced by $m^{-1}$, so that the variance of the estimate over all $m$ experiments

$$\text{var}_m \left( \frac{1}{c} \right) = \frac{\sigma^2}{m \Sigma x_i^2}$$

(4)

and therefore the precision of the estimation becomes

$$\text{precision} (m, \sigma, x_i) = \pm \sigma(m \Sigma x_i^2)^{-1/2}$$

(5)

This expression shows that the precision of the inverse speed of sound estimator may be improved by 1) increasing the number of experiments $m$; 2) using finer foam phantoms, thus achieving more homogeneity in the scattering properties of the material, which result in a reduced $\sigma^2$; and 3) increasing the length of each individual tracking interval so that the term $\Sigma x_i^2$ is increased.

Estimations in ultrasonically inhomogeneous tissue such as whole kidney resulted in precision which is almost an order of magnitude worse than those for the phantoms. In this case we used a combination of small $m$ (= 3), small $\Sigma x_i^2$ (due to the small size of the kidney), and large $\sigma^2$ (due to heterogeneity of the kidney (Fig. 8)). Thus (5) would predict worse performance than in the phantom case. Even though homogeneous tissues were not investigated, it is expected that such tissues will allow precision comparable to those for phantoms when a similar amount of data is available.

The signal processing scheme is an important parameter affecting $\sigma^2$. The peak detection method used here is probably suboptimal since it creates abrupt discontinuities in the data (Fig. 5–8) and thus an increased $\sigma^2$. An averaging method such as centroid detection or moving average could reduce $\sigma^2$ significantly.

In principle, the beam-tracking method would provide an unbiased accurate estimate of the speed of sound. This feature is difficult to assess experimentally, because the methods used to independently determine the absolute speed of sound in the materials under test do in themselves suffer from inaccuracies. The pulse–echo velocimeter method can be quite accurate, but significant error could be encountered when measuring the transit time of the pulse in the glycerol solution, which is a dissipative liquid with quadratic frequency dependent loss [23], [24].

This error is due to the narrowing of the spectral content and the downshift of the center frequency of the pulse. The accuracy of the beam tracking method might also suffer from a similar effect in as yet an unknown amount. The velocimeter measurement and the beam tracking estimations were nevertheless accurate to within ~ 1.2 percent of each other.

The effect of body wall refraction is investigated in the appendix. Under certain simplifying assumptions, notably two-dimensional treatment, small index of refraction and low amplitude/low spatial frequency shape of a single refractive boundary, it is shown that the local errors in the determination of the flight time of the pulse along the tracked beam are proportional to the second spatial derivative of the refractive boundary function, located beneath the tracking transducer. Therefore, a zeroth or first-order shape of the refractive layer (i.e., a parallel layer or a wedge-shaped layer) will have no effect on the precision or accuracy of the method. On the other hand, a second-order shape will bias the estimate. Additionally, the presence of an inclined refractive layer in conjunction with the tracked beam will cause bias errors, but no change in precision. If, however, many tracked beams are used and averaged, these bias errors might tend to cancel out. The data on the glycerol solution was obtained by tracking the beam through a 6.4 mm refractive acrylic wall at normal incidence. No effect which may be attributable to this wall is noted nor is it expected in the results.

Clutter in the received signals was noted in the porcine kidney experiment when the kidney was initially wrapped with 20 ppi open cell foam, which is a considerably stronger scatterer than the kidney itself. Off-axis or side-lobe echoes were strong enough to mask the desired signal such that no meaningful data were obtained. The removal of the foam allowed the reported data to be acquired.

The role of the transmitting and receiving transducers in the overall performance of the method was not investigated. It could be expected, however, that generally performance would improve when the effective volume of the beam intersection is decreased. The effective volume is related to the convolution of the transmitted pulse waveform and the effective cross-beam intensity profile of the receiving transducer. Thus narrow beamwidth and wide bandwidth could be desirable features of the measurement apparatus. On the other hand, the experimental results do not exhibit any particular degradation of the data in front or behind the focal region of the tracked-beam. This could be a result of 1) weak dependence of performance on the above parameters and 2) nonlinear (peak detection) processing that increases the "resolution" of the system, but which is also responsible for the stairstep like behavior of the data.

The stair-stepped appearance of the data is brought about in the following manner. When the tracking beam is incrementally moved along the tracked beam, one echo in the intersecting volume tends to dominate all others (Fig. 5). Several translational increments of the tracking beam are usually necessary before this dominant echo is
diminished and another echo becomes dominant. This explains the small plateaus in the data. Since peak detection is used, there usually is a point at which the peak algorithm ceases to recognize the previous echo peak and leaps abruptly (in a relatively large time increment) to recognize the newly dominant echo peak, thus creating a discontinuity in the data. This phenomenon is quite evident during the actual experiment. It is also interesting to note that the finer 30 ppi foam exhibits smaller, more numerous steps per unit length (compare Figs. 5 and 6), such that the approximate ratio between the number of steps per unit length and the porosity of the foam is maintained relatively constant (=0.2 steps/pore) regardless of which foam is used. Thus the appearance of the data is related to the scattering characteristics of the target. This relationship may in itself be useful for characterization of tissues.

It is evident that when the beams intersect at an angle of $\pi/2$, the effective volume of beam intersection at any given range is the smallest possible. Thus when angles that are either larger or smaller than $\pi/2$ are used, an effective increase in this volume will occur, which may cause degradation in performance. The extent to which this degradation occurs has not been investigated, but since a range of $\pi/4$ to $3\pi/4$ will result in a modest increase in the volume of intersection, it is expected that within this range the degradation would be small. Such a range of angles would allow greater flexibility in in vivo scanning.

Tissue movement during in vivo scanning might, in principle, become a problem, if the estimation were to be done in a slow fashion such as was done in the experiments reported in this paper. A much faster variation would use the dual array configuration with parallel receivers (Fig. 3), whereby tracking of one full echo-waveform would typically take only a fraction of a millisecond, thus essentially eliminating tissue movement during tracking.

## Appendix

### Two-Dimensional Error Analysis Due to Refractive Body Wall

We assume that refraction occurs along the path of the tracking beam due to a single layer of body wall fat, where the speed of sound in fat is constant and slightly lower than that of the liver. The liver is used here for illustrative purposes only, since it is large and homogenous. Other organs such as spleen may be suitable as well. From geometrical considerations (refer to Fig. 9):

\[
\alpha = \theta
\]

(1a)

and

\[
\alpha' = \theta'.
\]

(1b)

From Snell's law

\[
\frac{\sin \alpha}{\sin \beta} = \frac{\sin \alpha'}{\sin \beta'} = \frac{c_1}{c_2} = n
\]

(2)

where $n$ is the index of refraction, and $c_1$ and $c_2$ are the speeds of sound propagation in the fat layer and in the liver, respectively. The actual translation of the tracking-beam axis is $\Delta x$. In the absence of refraction, this would also be the translation of the beam inside the liver. Since refraction changes the orientation of the beam, the new translation becomes

\[
\Delta x' = \Delta x + \Delta - \Delta'.
\]

(3)

From Fig. 9

\[
\Delta = y \tan (\beta - \alpha) = y \tan (\beta - \theta)
\]

(4)

and

\[
\Delta' = y' \tan (\beta' - \alpha') = y' \tan (\beta' - \theta').
\]

(5)

We assume that $\theta$ and $\theta'$ are small angles, such that

\[
sin \theta \approx \tan \theta
\]

(6a)

and

\[
sin \theta' \approx \tan \theta'
\]

(6b)

This condition essentially means that only small-amplitude low-spatial-frequency roughness is considered. We further assume that the index of refraction is close to one, and thus angles $\beta, \beta'$ are small (a result of the small angle assumption for $\theta, \theta'$). These conditions must be imposed on the problem in order to keep the ensuing mathematics tractable. Significant departure from these constraints could obviously result in significantly different behavior from what is shown below.

From (4) and (5) we get under the previous conditions

\[
\Delta = y \sin (\beta - \theta)
\]

(7)

and

\[
\Delta' = y' \sin (\beta' - \theta').
\]

(8)

Rewriting in terms of trigonometric identities we get

\[
\Delta = y (\sin \beta \cos \theta - \cos \beta \sin \theta)
\]

(9)
and
\[ \Delta' = y'(\sin \beta' \cos \theta' - \cos \beta' \sin \theta'). \] (10)

Substituting in (3) we get
\[ \Delta x' = \Delta x + y(\sin \beta \cos \theta - \cos \beta \sin \theta) \\
- y'(\sin \beta' \cos \theta' - \cos \beta' \sin \theta'). \] (11)

From (1) and (2)
\[ \sin \beta = \frac{1}{n} \sin \theta \] (12)

and
\[ \sin \beta' = \frac{1}{n} \sin \theta'. \] (13)

Assuming again that the angles \( \theta, \theta', \beta, \beta' \) are small, and thus that all the cosine terms in (11) are \( \approx 1 \), we get from (11)
\[ \Delta x' = \Delta x + y\left( \frac{1}{n} \sin \theta - \sin \theta \right) \\
- y'\left( \frac{1}{n} \sin \theta' - \sin \theta' \right) \\
= \Delta x + \left( \frac{1}{n} - 1 \right) \left( y \sin \theta - y' \sin \theta' \right). \] (14)

Until this point we computed only the actual displacement of the beam intersections due to refraction. There are, however, additional small errors due to added travel time along the two tracking beams.

The travel time along the right-hand side (RHS) refracted path is
\[ \Delta t_{\text{RHS}} = \frac{y'}{c_2 \cos (\beta' - \theta')} \] (15)
and the travel time along the left-hand side refracted path is
\[ \Delta t_{\text{LHS}} = \frac{y}{c_2 \cos (\beta - \theta)} + \frac{y' - y}{c_1}. \] (16)

The travel time along the path \( \Delta x' \) is
\[ \Delta t_{\Delta x'} = \Delta x'/c_2. \] (17)

We now simplify the RHS of (15) as follows:
\[ \frac{y'}{c_2 \cos (\beta' - \theta')} = \frac{y'}{c_2(\cos \beta' \cos \theta' + \sin \beta' \sin \theta')}. \] (18)

Using the Snell law relationship \( \sin \beta' = (1/n) \sin \theta' \), we get
\[ \frac{y'}{c_2(\cos \beta' \cos \theta' + \frac{1}{n} \sin^2 \theta')} = \frac{y'}{c_2 \left[ \sqrt{(1 - \sin^2 \beta')(1 - \sin^2 \theta') + \frac{1}{n} \sin^2 \theta'} \right]}. \] (19)

The last term under the root \( \approx 0 \) for small \( \theta' \), so that,
\[ \Delta t_{\text{RHS}} = \frac{y'}{c_2 \left[ \sqrt{1 - \left( \frac{1}{n} \right)^2 \sin^2 \theta' + \frac{1}{n} \sin^2 \theta'} \right]}. \] (20)

Additionally, for small angle \( \theta' \), the second term under the root in (20) is \( \ll 1 \), and thus
\[ \Delta t_{\text{RHS}} = \frac{y'}{c_2 \left[ 1 - \frac{(n - 1)^2}{2n^2} \sin^2 \theta' \right]}. \] (21)

Similarly, the expression in equation (14) becomes
\[ \Delta t_{\text{LHS}} = \frac{y}{c_2 \left[ 1 - \frac{(n - 1)^2}{2n^2} \sin^2 \theta \right]} + \frac{y - y}{c_1}. \] (22)

and along the \( \Delta x' \) path the travel time is
\[ \Delta t_{\Delta x'} = \frac{1}{c_2} \left[ \Delta x + \left( \frac{1}{n} - 1 \right) \left( y \sin \theta - y' \sin \theta' \right) \right]. \] (23)

Using the relationship \( c_1 = nc_2 \), the difference in arrival times is
\[ \Delta t = \Delta t_{\Delta x'} + \Delta t_{\text{RHS}} - \Delta t_{\text{LHS}} \]
or
\[ \Delta t = \frac{1}{c_2} \left[ \Delta x + \left( \frac{1}{n} - 1 \right) \left( y \sin \theta - y' \sin \theta' \right) \right] - \frac{y - y}{n} + \frac{y'}{1 - \frac{(n - 1)^2}{2n^2} \sin^2 \theta'} \]
\[ - \frac{y}{1 - \frac{(n - 1)^2}{2n^2} \sin^2 \theta}. \] (24)

The inverse speed-of-sound estimator \( (1/c_2) = \Delta t/\Delta x \) is obtained by dividing (24) by \( \Delta x \), so that
\[ \frac{1}{c_2} = \frac{1}{c_2} \left\{ 1 + \left( \frac{1}{n} - 1 \right) \frac{y \sin \theta - y' \sin \theta'}{\Delta x} - \frac{(y' - y)}{n} \frac{1}{\Delta x} \right\} \]

\[ = \frac{y}{\Delta x \left[ 1 - \frac{(n - 1)^2}{2n^2} \sin^2 \theta \right]} \]

\[ + \frac{y'}{\Delta x \left[ 1 - \frac{(n - 1)^2}{2n^2} \sin^2 \theta' \right]} \].

Thus the inverse speed-of-sound estimate is equal to the actual inverse speed of sound plus four error terms.

We define the error terms as follows:

\[ \varepsilon_1 = \left( \frac{1}{n} - 1 \right) \frac{y \sin \theta - y' \sin \theta'}{\Delta x} \]  

(26)

\[ \varepsilon_2 = -\frac{1}{n} \left( \frac{y' - y}{\Delta x} \right) \]  

(27)

\[ \varepsilon_3 = \frac{-y}{\Delta x \left[ 1 - \frac{(n - 1)^2}{2n^2} \sin^2 \theta \right]} \]  

(28)

\[ \varepsilon_4 = \frac{y'}{\Delta x \left[ 1 - \frac{(n - 1)^2}{2n^2} \sin^2 \theta' \right]} \]  

(29)

When the index of refraction \( n = 1 \) (no refraction), the error term \( \varepsilon_1 = 0 \), the remaining errors \( (\varepsilon_3 + \varepsilon_4 + \varepsilon_4) = 0 \), and thus in this case the inverse speed of sound estimator is equal to the actual inverse speed of sound, as expected. The second term in the denominators of error terms \( \varepsilon_3 \) and \( \varepsilon_4 \) are small compared to one. For example, if \( n = 1.1 \) and \( \theta = 30^\circ \), this term equals \( 1.03 \times 10^{-3} \) and can be neglected. If this term is neglected, and for \( n = 1 \), the term \( \varepsilon_2 \) will tend to cancel the quantity \( (\varepsilon_3 + \varepsilon_4) \). Thus the only important error term remaining is \( \varepsilon_1 \).

For small \( \Delta x \) we can assume \( y = y' \), and thus

\[ \varepsilon_1 = \left( \frac{1}{n} - 1 \right) \frac{y (\sin \theta - \sin \theta')}{\Delta x} . \]  

(30)

If we again assume that angles \( \theta, \theta' \) are small and define \( \Delta \theta = \theta - \theta' \), then in the limit when \( \Delta x \) becomes very small we can write the local error as

\[ \varepsilon_1 \left|_{\Delta x \to 0} \right. = \lim_{\Delta x \to 0} \left[ y \left( \frac{1}{n} - 1 \right) \frac{\Delta \theta}{\Delta x} \right] = y \left( \frac{1}{n} - 1 \right) \frac{d \theta}{dx} \]  

(31)

for small \( \theta, \theta = \tan \theta \), and thus it equals the slope of the shape function \( s(x) \) of the boundary between the fat and the liver at the point where the tracking beam encounters the boundary, \( x_0 \), thus locally

\[ \theta = \frac{d s(x)}{dx} \bigg|_{x_0} . \]  

(32)

Differentiating both sides with respect to \( x \) we get

\[ \frac{d \theta}{dx} = \frac{d^2 s(x)}{dx^2} \bigg|_{x_0} \]  

(33)

and substituting in (31) we finally get the local error

\[ \varepsilon_1 \left|_{\Delta x \to 0} \right. = y \left( \frac{1}{n} - 1 \right) \frac{d^2 s(x)}{dx^2} \bigg|_{x_0} . \]  

(34)

This is an important result from which the following observations can be made regarding the behavior of the local error.

1) The local error is proportional to the second derivative of the shape of the refracting boundary. If we model the refracting boundary in a general Taylor series

\[ s(x - x_0) = A_0 + A_1 (x - x_0) + A_2 (x - x_0)^2 + \cdots \]  

(35)

it is clear from (34) that a second order and higher boundary shape will introduce errors in the estimate. Zeroth-order (planar) and first-order (wedge shape) boundaries will cause no refractive errors, as long as the constraint of small \( \theta \) which was imposed in the derivation is maintained. This is also intuitively satisfying since the zeroth order boundary causes no refraction, and the first order boundary causes equal bending of all tracking beams, and thus no net error.

2) A given refractive local error is directly amplified by the distance \( y \) from the boundary to the point where the beams intersect. It is thus desirable to make the estimations in regions that lie just beneath the refracting boundary, in order to minimize the error.

**ACKNOWLEDGMENT**

The author acknowledges the support received from Dr. A. Gobuty and Ms. H. Aquino in the preparation of the biological specimen.

**REFERENCES**


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