ON THE FEASIBILITY OF PULSE-ECHO SPEED OF SOUND ESTIMATION IN SMALL REGIONS: SIMULATION STUDIES

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Abstract—Computer simulations are used to study the feasibility of the estimation of sound speed in small regions with precision better than 1% using the Beam Tracking method. The speed of sound is estimated in a 10-mm by 10-mm region by considering a number of parallel tracks confined to the small region. The transducer focusing and the step sizes for the tracking and tracked transducers required to extract the maximum amount of uncorrelated data from the 10-mm by 10-mm region is evaluated. The results show that the speed of sound can be estimated with error less than 1% in a small region using a typical medical transducer. The statistical comparison of estimates in small areas with different speed of sound is also considered.

Key Words: Sound speed, Sound speed contrast, Beam Tracking method, Simulation, Soft tissue, Decorrelation distance, Beamwidth, Ultrasound, Crosscorrelation.

1. INTRODUCTION

In recent years, several methods for the estimation of sound speed in pulse-echo mode have been proposed. Motivation for these studies arises from an apparent correlation between the sound speed and tissue pathology. For example, a change in the sound speed of a few percent is expected in some medically significant situations in the liver (Lin et al. 1987). Robinson et al. (1982) and Bamber and Abbott (1985) developed techniques that rely on the presence of a discrete target; the amount of displacement between two images of the target obtained from two directions is related to the discrepancy between the true speed of sound and the speed of sound which is assumed by the scanner. A similar technique involving a discrete target is due to Hayashi et al. (1985) and Katakura et al. (1985), whereby the quality of transducer array focusing is maximized by interactive user control of signal delays at the aperture. These authors have reported a ±2% precision in liver in vivo. Ohtsuki et al. (1985) have reported the Reference Point Technique which uses two reference points, such as the edges of a tumor. The actual distance between the reference points is measured along one axis using a linearly scanned image, while the time of flight in the tissue contained between the reference points is measured from an A-mode signal obtained from a transducer at right angles. They report an accuracy error in their measurement of about 10%. Iinuma et al. (1985) and Akamatsu et al. (1985) have used cross beam methods to measure the total transit time of the pulse and the echoes produced from diffuse scatterers. A ray tracing method for calculating the speed of sound in a non-parallel layered model was introduced by Lockwood et al. (1989). Ophir (1986) has reported a Beam Tracking method resulting in accurate and precise estimates (<1% error) of speed of sound in diffusely scattering phantoms and in tissue in vitro. The optimization of the signal processing (Ophir et al. 1989a) and statistical considerations (Kontonasios and Ophir 1986) for Beam Tracking have been studied. The effect of correlation artifacts (Ophir et al. 1989b), and the correction for refraction and other angle errors using multiple tracking transducers (Shattuck et al. 1989) have also been discussed.

Most of these methods estimate the average speed of sound in large regions of interest, such as in whole organs. They are therefore suitable for diagnosis of diffuse disease. However, it may also be medically advantageous to obtain more localized estimates to assess the characteristics of the tissue in different regions within an organ. Ultimately, a speed of sound image could be medically useful.
Beam Tracking allows local measurement of the sound speed, but so far the method has only been applied to single, relatively long tracks (75 mm), achieving precision errors on the order of 0.1% (Ophir et al. 1989a). Since the expected changes in the speed of sound due to tissue pathology are on the order of a few percent (Bamber and Hill 1981; Chen et al. 1987; Lin et al. 1987), this form of Beam Tracking provides us with satisfactory precision, but the long tracks limit its usefulness to diffuse disease in large organs. In theory, similar precision can be achieved from a number of shorter, parallel tracks confined to a small region of tissue. Thus, the objective of this work is to determine the feasibility of the estimation of sound velocity in a small region of interest, arbitrarily defined as a 10-mm by 10-mm area, to within 1%. The measurement error is to be optimized by selecting appropriate transducers, and by the proper application of the Beam Tracking method.

A simulation approach was used in this work due to the practical difficulties of changing transducer characteristics and making measurements which are independent of other system variables. In previous work, we have shown good agreement between Beam Tracking simulation and experiment (Ophir et al. 1989a). An expanded version of the simulation is used, in which we introduce a two-dimensional transducer model.

Before describing the estimation in small regions, the Beam Tracking method is reviewed to provide background. Then, the computer model which simulates Beam Tracking is described. The simulation program is used to determine the optimal working conditions of Beam Tracking for estimating sound speed in a small region. Finally, the limitations and trade-offs of the method are discussed.

2. THE BEAM TRACKING METHOD

The Beam tracking method has already been presented (Ophir 1986) and will not be discussed in detail in this paper. Only a basic description of the method and the main variables which affect its precision are given below.

Two coplanar transducers are positioned such that their beams intersect at a given angle in a volume of diffusely scattering tissue. One (tracked) transducer serves as a transmitter whose beam is tracked by the second (tracking) transducer which serves as a receiver. Initially, the receiving transducer is positioned at some arbitrary position R1 (see Fig. 1). Some sound energy is scattered in the volume of intersection of the two beams V1 where a large number of randomly distributed small scatterers is assumed to exist. The received signal resembles a noisy Gaussian modulated carrier which is unsuitable for direct time delay measurement. Upon full wave rectification and low-pass filtering, the signal is used to estimate the time of flight \( t_1 \) of the pulse from the transmitter to V1, and of the echoes to the receiver.

The tracking transducer is then moved in a direction parallel to the tracked beam to a new location R2, which is separated from its previous position by a distance \( \Delta x \). The new time of flight through V2 is now \( t_2 \). This process is repeated for \( i \) locations of the tracking transducer and \( i \) values of \( t \) are obtained. A plot of time vs. position is made, and a least squares linear regression fit is performed, where the slope of the fit is \( \hat{\beta} = 1/\hat{c} \) and \( \hat{c} \) is the estimate of the speed of sound. Additional parallel tracks are obtained by moving the tracked transducer in incremental steps \( \Delta y \) (Fig. 1).

We have shown (Kontonassios and Ophir 1986) that the standard deviation (SD) of the estimate \( \hat{\beta} \) is given by

\[
\text{SD}(\hat{\beta}) = \pm 2 \left( \frac{3 \sigma^2 \Delta x}{m \lambda^3} \right)^{1/2},
\]

where \( \sigma^2 \) is the estimated variance of the residuals of the fit obtained by subtracting the linear trend from the arrival times, \( \lambda \) is the total length of one track, and \( m \) is the number of uncorrelated tracks (for which \( \Delta y \) is greater than the correlation length between tracks). It can be shown (see Appendix B) that the SD of \( \hat{c} \) is proportional to the SD of \( \hat{\beta} \), viz.

\[
\text{SD}(\hat{c}) = \hat{c}^2 \text{SD}(\hat{\beta}).
\]

Thus, eqns (1a) and (1b) show that the speed of sound estimator may be improved by increasing the number of uncorrelated tracked beams, increasing
the length of the track, decreasing the step size and/or decreasing the variance of the residuals. Although the first two factors above improve the precision, they require a larger tissue region of interest. Decreasing the step size Δx improves the precision but only as long as the decorrelation of the samples is maintained. A lack of such a decorrelation results in stairstep artifacts which introduce bias and precision errors (Ophir et al. 1989b). Finally, reducing the variance of the residuals is a desirable way to improve the precision, which can be done by proper choice of the estimators and appropriate filtering (Ophir et al. 1989a). It is assumed in this description of Beam Tracking that there are no bias errors due to refraction in the body wall. Correction for errors of this nature are discussed by Shattuck et al. (1989).

In this work we estimate the speed of sound in a small square region of interest by reducing the length of the tracks and increasing the number of parallel tracks confined to a small region. The reduction of the track length results in rapid deterioration of the precision of the estimate as shown by eqn (1a). To compensate for this deterioration, it becomes necessary to increase the number of uncorrelated tracks in the region of interest. However, this process is inefficient; for example, 8 uncorrelated tracks are needed to maintain the original precision (keep the product ml² constant) if we reduce the track length by half (eqn 1a).

The maximization of the number of uncorrelated tracks involves the optimal selection of transducer beam characteristics. For a given transducer, uncorrelated tracks can certainly be obtained by moving the tracked transducer by a whole beamwidth (Wagner et al. 1983). A larger number of uncorrelated tracks can be obtained by either reducing the beamwidth of the tracked transducer and/or by moving the tracked transducer by a fraction of its beamwidth (this issue will be discussed later). Note, however, that for a given operating wavelength (see Appendix A) a reduction of the beamwidth results in a rapid reduction of the focal depth. Since we want to track within the narrow region of the beam (i.e., the focal region), reducing the tracked beamwidth may limit the practical length of the track. Therefore, the advantage of reducing the tracked beamwidth is constrained.

3. SIMULATION

The simulation studies are performed using a program that models the transducer, the scattering tissue, and the Beam Tracking method in a two-dimensional setup.

3.1. Transducer model

The transducer is modeled as an array of closely spaced aperture elements and is defined by the following specifications which are part of the input data to the program:
- spacing of the aperture elements (not to exceed λ/2),
- aperture size,
- center frequency (f₀),
- bandwidth (BW), and
- focal distance (Z₀).

The impulse response of the transducer at any point in the sound field is computed by adding up the individual contributions of the elements of the array (Céspedes and Ophir 1990). The elements are considered to be ideal point sources/receivers, having a Gaussian transfer function with center frequency (f₀) and bandwidth (BW). A fully sampled aperture is achieved by selecting the spacing between aperture elements to be λ/2 (smaller spacing improves accuracy at a significant computational cost). By appropriately delaying the contribution of each element, the transducer is focused at distance Z₀. Thus, the simulation of different transducers is achieved by changing the input parameters. Note that a factor of 1.08 must be applied to the size of the aperture of our one-dimensional line transducer to represent the field of an equivalent circular aperture (Céspedes and Ophir 1990; Schade 1964; Wagner et al. 1983).

3.2. Tissue model

A non-attenuating tissue sample with speed of sound c is modeled by means of a set of randomly located point scatterers. The set of scatterers is represented by a 400 by 400 character array, where an asterisk denotes the presence of a scatterer and a blank denotes the absence of a scatterer at any given location. The set of scatterers corresponds to an area A of user-specified dimensions, and therefore the density of scatterers (# of scatterers/A) is a variable parameter. Typically, the whole array represents a 10-cm by 10-cm area with scatterer density of approximately 20 mm⁻². A square region with a different speed of sound than that of the background can be defined by specifying the coordinates of its lower-left corner and upper-right corner.

3.3. Beam Tracking model

The program simulates a Beam Tracking experiment, taking as inputs:
- the specifications for the transmitting transducer and the receiving transducer,
2. the position of the transducers, the corresponding step sizes $\Delta x$ and $\Delta y$, and
3. the file containing the simulated tissue model.

For each intersection region of the transmitting and receiving transducer beams, the program constructs a radio frequency (RF) echo sequence, taking into account the reflections from all the scatterers in that region. All scatterers are considered to have the same scattering cross-section, and multiple scattering and attenuation are ignored. The area of intersection of the beams is defined by the theoretical beamwidth at the focus measured between the first lateral field intensity nulls of each transducer operating at the center frequency $f_0$. The RF signal is demodulated and filtered as described in Ophir et al. (1989a) to obtain an optimum estimate of the time of flight, and the corresponding travelled distance is computed based on the positions of the transducers. The output of the program is a table of the times of flight vs. transducer positions, which is stored for further processing. A simple linear regression algorithm is then used to obtain the speed of sound estimate.

4. METHODS

4.1. Estimation in a small area

In the Beam Tracking method, the tracking transducer takes incremental steps of size $\Delta x$, while the tracked transducer remains stationary. As an extension to this method, the estimation in a small region of interest is done by averaging sound speed estimates from a number of short parallel tracks within that region. For this purpose, the tracked transducer takes incremental steps $\Delta y$. Since we have limited the length of all tracks to 10 mm and the variance of the residuals is minimized by the signal processing method utilized (Ophir et al. 1989a), special attention is given to the maximization of the number of uncorrelated tracks $m$ and to the minimization of the uncorrelated step size $\Delta x$ within the 10-mm by 10-mm region of interest (eqn 1a) which result in further improvement to the precision of the estimation.

By allowing $\Delta x$ and $\Delta y$ to be smaller than the width of the corresponding beams, the number of intersections per track and the number of tracks per region can be increased. In this way, consecutive tracks and consecutive intersections of the beams within one track interrogate the tissue through spatial windows which overlap. The steps $\Delta x$ and $\Delta y$ should be the smallest possible, and yet satisfy the condition that the data obtained from adjacent tracks and intersections remain uncorrelated. The minimum $\Delta x$ and $\Delta y$ step sizes which satisfy this condition are termed the decorrelation distances.

There is, however, a physical limitation to the smallest useful values of $\Delta x$ and $\Delta y$. As reported in Ophir et al. (1989b), overlapping the beam intersections along the track beyond the decorrelation distance results in stair-step artifacts in the measurements, which cause an increase in the variance of the speed of sound estimate within that track. Excessive overlapping between tracks results in correlated data which does not contribute to the reduction of randomness through averaging. The optimal selection of the step sizes $\Delta x$ and $\Delta y$ is therefore an important issue.

4.2. Determination of decorrelation distances

For a given transducer, estimates of the appropriate decorrelation distances along the beam and across beams are made. First, a straight line is fitted to the time of flight vs. distance curve using the least squares criterion, and the resulting linear trend is subtracted from the data. The resultant random process $r(x)$ depicts the residuals of the fit. The autocorrelation of the residuals is

$$R_s(\tau) = \sum_i r(x_i)r(x_i + \tau). \quad (2)$$

The minimum step for the tracking transducer that produces uncorrelated data, termed the decorrelation distance along the beam, $\Delta x_{\text{min}}$, is defined by the $-20$ dB width of the normalized autocorrelation function

$$\tilde{R}(\tau) = R_s(\tau)/R_s(0), \quad (3)$$

that is,

$$\Delta x_{\text{min}} = [\tilde{R}(\tau)]_{-20\text{dB}}. \quad (4)$$

Figure 2 shows a typical normalized autocorrelation function along the beam.

![Fig. 2. Typical normalized autocorrelation of the residuals of the linear fit to the times of arrival.](image)
The crosscorrelation of the residuals of two adjacent tracks, designated 1 and 2, is given by

\[ R_{12}(\tau) = \sum_i r_i(x_i) r_2(x_i + \tau), \] (5)

where \( r_i \) are residual functions and the subindices in the functions indicate track numbers. Similarly, the minimum step for the tracked transducer is termed the decorrelation distance across beams, \( \Delta y_{\min} \); it is defined by the \(-20\) dB width of a function whose values are the normalized crosscorrelations at \( \tau = 0 \) between the first track in a region of interest and successive tracks in that region, i.e.,

\[ \Delta y_{\min} = [R_{11}(0), R_{12}(0), R_{13}(0), R_{14}(0), \ldots]_{-20\text{dB}} \] (6)

where the subindices of the crosscorrelation functions indicate track numbers.

4.3. Simulations

To assess the relationship between the decorrelation distances and the transducer beamwidth, \( \Delta x_{\min} \) and \( \Delta y_{\min} \) have been measured for a number of transducers with different apertures. All parallel tracks are obtained from within the 3-dB focal depth of the tracking transducer and the tracking is performed within the focal depth of the tracked beam. We simulated standard piston transducers with diameters ranging from 15 mm to 87 mm focused at 100 mm with an operating frequency 3.5 MHz and 60\% two-way \(-3\) dB bandwidth. The simulated sampling frequency was 50 MHz.

To verify the existence of optimal step sizes, the SD of the speed of sound estimates in a 10-mm by 10-mm region was computed for a range step sizes of the tracked and tracking transducers. With a fixed tracked step size of \( \Delta y = 1 \) mm, the precision of the speed of sound estimate was computed for tracking step sizes ranging from 0.25 mm to 3.3 mm. With a fixed tracking step size of \( \Delta x = 1 \) mm, the precision of the speed of sound estimate was computed with the number of tracks within the 10-mm by 10-mm area ranging from 3 (\( \Delta y = 3.3 \) mm) to 50 (\( \Delta y = 0.2 \) mm). These simulations were performed with 44-mm transducer apertures with \(-3\) dB beamwidth of 1 mm, and a \(-3\) dB focal length of 16 mm (see Appendix A).

Finally, simulations were performed for small regions with true speeds of sound which were 1\%, 2\%, and 3\% higher than the background value of 1500 m/s. The objective of these simulations was to determine the ability of this method to discriminate in a statistically significant manner between regions with small speed of sound “contrast” as a function of the region size. These simulations used 1 mm \(-3\) dB beamwidth transducers and step sizes \( \Delta x = \Delta y = 0.5 \) mm. In all simulations, the tracking and tracked transducers were taken to have the same characteristics.

5. RESULTS

Figure 3 shows the decorrelation distance \( \Delta x_{\min} \) in mm (eqn 4) as a function of the tracking beam-
Fig. 4. Decorrelation distance between tracks vs. tracked beamwidth.

width. Figure 4 shows the decorrelation distance $\Delta y_{\text{min}}$ (eqn 6) as a function of the tracked beamwidth.

The precision of the speed of sound measurements is described here by the SD of the estimates. Figure 5 shows theoretical (eqn 1b) and measured SD of the speed of sound estimates for a 1 mm $-3$ dB beamwidth with a constant number of tracks ($m = 10$) and different tracking step sizes. The precision of the speed of sound estimate improves with decreasing tracking step size, reaching a minimum deviation of $7.5 \text{ m/s} (0.5\%)$ for $\Delta x \approx 0.5 \text{ mm}$ and remaining approximately constant thereafter. Figure 6 shows the theoretical (eqn 1b) and measured SD of the speed of sound estimate for a fixed tracking step size ($\Delta x = 1 \text{ mm}$) and different number of tracks. This graphs show an improvement of the precision for increasing number of tracks reaching a minimum deviation of $\approx 5 \text{ m/s} (.33\%)$ for $m = 40$ (i.e., $\Delta y \approx 0.25 \text{ mm}$) and remaining constant thereafter.

Figure 7 presents the statistical significance of the difference between estimates from regions with different speed of sound contrast (parametric resolution).

Fig. 5. Theoretical and measured SD of speed of sound estimates vs. tracking step size in a 10-mm by 10-mm region; 10 parallel tracks are considered.
Fig. 7. Level of statistical significance of the difference between estimates obtained from scanned regions with 1%, 2% and 3% true difference in their speeds of sound from the surround. The resolution is a linear measure of the size of the square regions of interest. The dashed line corresponds to extrapolated data.

as a function of region size (spatial resolution). The study was done with a modified t-test for small samples as described in Lin and Ophir (1987). The dashed line in the 1% speed of sound contrast case represents extrapolated values for regions larger than 10 mm by 10 mm.

6. DISCUSSION AND CONCLUSION

The results of simulations show that there are optimal tracking and tracked step sizes above which the information available in a region of tissue is not efficiently extracted, and below which the estimation error no longer improves. This is illustrated in Figs. 5 and 6 where for a 1-mm beamwidth the estimation error does not decrease further for tracked step sizes lower than \( \approx 0.25 \text{ mm} \), and for tracking step sizes lower than \( \approx 0.5 \text{ mm} \). Figures 5 and 6 show a good overall match between the theoretical and simulation results. Note, however, that eqn (1a) is valid for uncorrelated data only; the simulation and theory differ for small step sizes (\( \Delta x \leq 0.5 \text{ mm}, \Delta y \leq 0.25 \text{ mm} \)) which do not satisfy the decorrelation requirement.

For highly focused transducers with small beamwidths, care must be taken to keep the region of interest confined within the focal depth of the transducers. This maximum size of the region of interest is constrained by the tracking and tracked focal depths.

The decorrelation distances are linearly related to the 3-dB beamwidth of the transducer as seen in Figs. 3 and 4. The proportionality constants derived from the curves are given by

\[ \Delta y_{\text{min}} \approx 0.75 \, dr(3 \text{ dB}), \]

and

\[ \Delta x_{\text{min}} \approx 0.55 \, dr(3 \text{ dB}), \]

where \( dr(3 \text{ dB}) \) is the \(-3 \text{ dB} \) beamwidth. However,
decoration appears to occur at a fraction of the 20-
3 dB decorrelation distances \( \Delta y_{\text{min}} \) and \( \Delta x_{\text{min}} \). For ex-
ample, with the 1 mm – 3 dB beamwidth transducer,
decoration along the track occurs at \( \approx 0.9 \Delta x_{\text{min}} = 0.5 \text{ mm} \) (Fig. 5), and decoration between tracks occurs at approximately \( m = 40 \) or 0.33 \( \Delta y_{\text{min}} = 0.25 \text{ mm} \) (Fig. 6).

The statistical comparison of the estimates in small areas with different speeds of sound shows that it is possible to differentiate at the \( P_r < 0.001 \) level of significance between two 8.5-mm by 8.5-mm simulated tissue samples with known 2% differences in their speeds of sound. For regions with 1% difference, an area of 11.5 mm by 11.5 mm is needed for the same level of significance. For smaller regions, the level of significance of the difference between regions increases with the area as shown in Fig. 7. For a given probability \( (P_r) \) value, larger areas are needed to detect smaller differences in speed of sound and vice versa.

In conclusion, this simulation study shows that the selection of the transducers and the corresponding step sizes are critical in further reducing the estimation error for a given area of tissue. Once the optimal transducers and step sizes are selected, there is an expected trade-off between spatial and parametric resolution. We have shown that it is possible to differentiate between small areas possessing small differences in speeds of sound using a typical medical transducer. Thus, in principle we can map speed of sound estimates in small regions to pixels in a gray-scale image, where the gray level indicates the magnitude of the estimated speed of sound. For statistically significant differentiation between regions differing by the 1% criterion, regions on the order of 10 mm by 10 mm are required when optimal transducers are employed.

This simulation assumes the ideal situation of a non-attenuating medium and uses a continuous-wave approximation to the beam dimensions (see Appendix A). In practice, deviations may be expected from the quantitative results presented here and it may be necessary to compensate for the non-ideal situation. Nevertheless, we expect our conclusions to be generally valid.

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tion of refraction and other angle errors in Beam Tracking speed of sound estimation using multiple tracking transducers. Ultra-

APPENDIX A

In the ideal case of a non-attenuating medium and continuous wave, the \(-3\) dB focal spot size of a circularly symmetric transducer is given as (Kino 1987)

\[
dr(3\,\text{dB}) = 1.02\lambda f,
\]

and the \(-3\) dB depth of focus is given by

\[
dz(3\,\text{dB}) = 7.1\lambda f^2,
\]

where \(\lambda\) is the wavelength and \(f\) is the \(F\) number.

For example, for a transducer with diameter \(d = 44\) mm, focused at \(Z_o = 100\) mm, and center frequency \(f_o = 3.5\) MHz, in a medium with \(c = 1500\) m/s, we have: \(\lambda \approx 0.43\) mm, \(f = Z/d = 2.2\), \(dr(3\,\text{dB}) \approx 1\) mm, and \(dz(3\,\text{dB}) \approx 16\) mm. The focal depth and beamwidth can be changed by varying either \(\lambda\) or \(f\). Note, however, that changing \(\lambda\) will change \(dr(3\,\text{dB})\) and \(dz(3\,\text{dB})\) proportionally, but changing \(f\) has a stronger effect on \(dz(3\,\text{dB})\). Therefore, care must be exercised in choosing the proper transducer for optimal speed of sound estimation.

APPENDIX B

Given \(\hat{\beta} = 1/\hat{c}\), a deviation in \(\hat{\beta}\) results in a deviation in \(\hat{c}\) as follows:

\[
\hat{\beta} + \delta\hat{\beta} = 1/(\hat{c} - \delta\hat{c}),
\]

where \(\hat{c}\) is the estimated speed of sound.

Dividing both sides by \(\hat{\beta}\),

\[
1 + \delta\hat{\beta}/\hat{\beta} = \hat{c}/(\hat{c} - \delta\hat{c}).
\]

Assuming that \(\delta\hat{c} \ll \hat{c}\), we get

\[
\delta\hat{\beta}/\hat{\beta} \approx \delta\hat{c}/\hat{c},
\]

and thus

\[
\delta\hat{c} \approx \delta\hat{\beta}/\hat{c}^2,
\]

or

\[
\delta\hat{c} = \hat{c}^2\delta\hat{\beta}.
\]

This equation shows that the precision of \(\hat{c}\) is linearly related to the precision of \(\hat{\beta}\), with constant of proportionality \(\hat{c}^2\).