METHODS FOR ESTIMATION OF SUBSAMPLE TIME DELAYS OF DIGITIZED ECHO SIGNALS

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Time delay estimation (TDE) is commonly performed in practice by crosscorrelation of digitized echo signals. Since time delays are generally not integral multiples of the sampling period, the location of the largest sample of the crosscorrelation function (ccf) is an inexact estimator of the location of the peak. Therefore, one must interpolate between the samples of the ccf to improve the estimation precision. Using theory and simulations, we review and compare the performance of several methods for interpolation of the ccf. The maximum likelihood approach to interpolation is the application of a reconstruction filter to the discrete ccf. However, this method can only be approximated in practice and can be computationally intensive. For these reasons, a simple method is widely used that involves fitting a parabola (or other curve) to samples of the ccf in the neighborhood of its peak. We describe and compare two curve-fitting methods: parabolic and cosine interpolation. Curve-fitting interpolation can yield biased time-delay estimates, which may preclude the use of these methods in some applications. The artifactual effect of these bias errors on elasticity imaging by elastography is discussed. We demonstrate that reconstructive interpolation is unbiased. An iterative implementation of the reconstruction procedure is proposed that can reduce the computation time significantly.


Keywords: Bias error; Cramér-Rao lower bound; curve fitting; elastography; interpolation; signal reconstruction; time delay estimation; ultrasound.

I. INTRODUCTION

A large part of the literature on time-delay estimation (TDE) arises from sonar and radar research and is mainly concerned with signal detection in low signal-to-noise ratio (SNR) environments (SNR < 0 dB). Many of the methods proposed are compared against the theoretical lower bound of the precision of time-delay estimators known as the Cramér-Rao lower bound (CRLB) [1]. Quazi [2] compiled expressions for CRLB for high and low SNRs, and for low-pass and band-pass signals.
Since in this paper we are concerned with correlation processing of ultrasonic radiofrequency (rf) echo signals, we are mostly interested in the band-pass case, and in particular in the high SNR situation present in applications such as elastography [3]. For band-pass stationary signals with rectangular spectra and in the presence of low-level, uncorrelated, additive white noise (SNR $\gg$ 0 dB), the CRLB is given by

$$\sigma_{\text{CRLB}} \equiv \frac{1}{2\pi f_0 \sqrt{BT \sqrt{\text{SNR}} / (1 + B^2 / 12f_0^2)}}. \quad (1)$$

where $T$ is the signal duration (observation time), $f_0$ is the center frequency, $B$ is the signal bandwidth, and SNR is defined as the ratio of the (in-band) powers of signal and noise [2, Eq. (23)]. In ultrasonic imaging, where the fractional bandwidth is typically 50 percent to 80 percent, the last term in the denominator of Eq. (1) can be ignored and the resulting expression for the CRLB is

$$\sigma_{\text{CRLB}} \equiv \frac{1}{2\pi f_0 \sqrt{BT \sqrt{\text{SNR}}}}. \quad (2)$$

In medical ultrasound, Gaussian-shaped spectra, rather than the rectangular spectrum used to derive Eqs. (1) and (2), are usually used to model the signal spectral characteristics. However, we demonstrate in Appendix A that these expressions are also approximately valid for Gaussian spectra, provided that we replace the signal bandwidth $B$ with the equivalent noise bandwidth of the Gaussian. The equivalent noise bandwidth is the width of a rectangular spectrum having the same power and peak amplitude as the true, in this case Gaussian, spectrum. For a Gaussian spectrum given by

$$S(f) = e^{-f^2/2\sigma^2} \quad (3)$$

the equivalent noise bandwidth is $B_e = \sqrt{2\pi \sigma} \approx 2.5 \sigma$, where $\sigma$ is the standard deviation of the power spectrum [see Appendix B]. It has been noted that the validity of using the equivalent noise bandwidth in Eq. (2) can be good for many spectra [4], but is of course not exact.

Thus, the precision of TDE of analog signals is mainly determined by the center frequency, bandwidth, observation time, and SNR of the signal (Eq. (2)). However, when processing sampled signals, additional considerations apply regarding time and amplitude quantization and their impact on the time-delay errors. These errors, which can be much larger than those predicted by Eq. (2) at high SNR, limit the relevance of the Cramér-Rao lower bound in interpreting the real performance of time-delay estimation using sampled signals [5].

In the crosscorrelation approach to TDE, the location of the peak of the crosscorrelation function (ccf) is the estimate of the time delay. Implicit in most digital crosscorrelation TDE studies is that the errors due to finite sampling are insignificant compared to noise-related time-delay errors. This is a reasonable assumption in low SNR systems where the noise-related time-delay errors can be much larger than those related to time quantization. However, in practice, the time delays are generally not integral multiples of the sampling interval. This situation is more
relevant in high SNR systems where the time-delay errors due to time quantization can be dominant.

Thus, in general, the location of the largest sample of the ccf is an inexact estimator of time delay. The time-delay error due to quantization ($\epsilon_q$) is in the range $-Ts/2 \leq \epsilon_q \leq Ts/2$, where $Ts$ is the sampling period. For a small $Ts$, it is reasonable to assume that $\epsilon_q$ is a random variable uniformly distributed from $-Ts/2$ to $Ts/2$. Further assuming that successive time-delay errors are uncorrelated with each other and with the time-delay, the standard deviation of the time-delay error due to time quantization only is

$$\sigma_q = \frac{Ts}{\sqrt{12}}$$

[6, p. 51]. As an example, we consider a typical medical ultrasound signal propagating in lossless tissue, and which has a Gaussian spectrum (with center frequency $f_0 = 5$ MHz; standard deviation $\sigma = 0.3 \times f_0 = 1.5$ MHz, $B_n = 3.75$ MHz), sampled at 50 MHz ($\Delta T = 20$ ns), with an observation depth of 5 mm (observation time $T = 6.7$ $\mu$s), and additive -40 dB white noise. For such a case we obtain the following Cramér-Rao lower bound and quantization errors (using Eqs. (2) and (4), respectively),

$$\sigma_{CRLB} \equiv 0.024 \text{ ns} << \sigma_q = 5.7 \text{ ns}.$$  (5)

It is important to note that the SNR in Eq. (2) is the SNR in the bandpass of the signal and that the signal is usually only present in a fraction of the frequency band (in this example 3.75 MHz out of 50 MHz/2 = 25 MHz). In our example, the in-band noise power is 25 MHz/3.75 MHz = 6.6 times smaller than the total noise power which is 40 dB lower than the signal power.

The above discussion of precision errors and the numerical example demonstrate that time-delay estimation errors due to quantization can dominate the achievable time-delay precision in high SNR situations. Applications that require time delay precision better than that imposed by time quantization, must use interpolation of the samples of the digital ccf to precisely locate the peak [5, 7, 8]. Certainly, time-delay errors in low SNR situations can be on the order of or larger than the errors introduced by time quantization, in which case ccf interpolation may become irrelevant; these situations are not addressed in this paper.

As discussed later, some ccf peak interpolation techniques yield biased estimates of time delay. In this paper, we review and compare the performance of three methods for interpolation of the ccf. The CRLB is used as a benchmark for the performance of crosscorrelation time-delay estimators; however, the bias errors of the different interpolation methods, rather than the precision errors, are the main focus of this paper.

The maximum likelihood estimation of subsample time delays can be shown to be the application of a reconstruction filter to the discrete samples of the ccf peak [5]. However, estimating time delay by this method can only be approximated in practice and can be computationally intensive. For these reasons, a simple method is
widely used that involves fitting a parabola (or other curve) to samples of the ccf in the neighborhood of its peak. Although computationally simple, curve-fitting methods are generally biased; the bias error is a function of the subsample part of the time delay [5]. This may preclude the use of curve-fitting methods in some high SNR applications and therefore reconstructive interpolation may be the method of choice in those cases.

In medical ultrasound, time-delay estimation is used in several applications. Subsample time-delay resolution is required in elasticity imaging techniques such as elastography [9, 10] that operate under high SNR conditions. For typical parameters, the standard deviation of elastographic image noise obtained from TDE without interpolation is approximately 100 percent of the mean image amplitude [10]; such level of image noise severely limits the quality of the images. Crosscorrelation interpolation methods have also been studied for blood flow measurement techniques [11-14], where the SNR may be significantly lower than in elastography, and for phase aberration correction techniques [15-17].

We investigate two kinds of methods for ccf peak interpolation: curve-fitting methods and reconstructive methods. Specifically, we discuss the interpolation of the discrete-time (time-quantized) ccf, leaving the study of amplitude quantization effects for future studies (the quantization noise can also be considered as a source of noise). Two curve-fitting methods are reviewed: the polynomial and the cosine interpolation methods; the bias errors of these methods are analyzed, and their artifactual effect on elastographic imaging is discussed. Three reconstructive interpolation methods are discussed: the zero-padding method, reconstructive filtering method, and iterative reconstructive filtering method. We demonstrate that reconstructive interpolation is unbiased and that it can achieve the CRLB.

II. METHODS

The crosscorrelation function between two signals $y(t)$ and $x(t)$ is given by

$$R_{xy}(\tau) = E[y(t)x(t+\tau)],$$

(6)

where $E[\cdot]$ denotes expectation. The argument $\tau$ that maximizes the ccf provides an estimate of time delay. Because of the finite observation time available in practice, however, the crosscorrelation function can only be estimated. In this work, we estimate the crosscorrelation function with the crosscorrelation coefficient function [6]; the crosscorrelation coefficient function approach has been used in the past by several authors [18-20].

In order to investigate the different methods to interpolate the ccf, we used simulation and theoretical studies.

A. Simulation

The simulation of echo signals used in this paper has been described in detail elsewhere [10, 21]. In brief, the simulation is composed by models for the transducer and scattering medium. The transducer is modeled as a one-dimensional array of point elements; each element has a Gaussian transfer function. The spacing between
elements is $\lambda/2$, where $\lambda$ is the wavelength in the target associated with the center frequency of the transducer. The scattering medium is simulated by a 2D collection of scatterers with random position, amplitude and polarity; all scatterers have identical scattering cross section, and multiple scattering and attenuation are ignored. Simulated echo signals are constructed by adding up the impulse responses of the transducer at the locations of all scatterers (in a defined region of interest) with consideration for appropriate travel times and beam spreading effects. The simulation, which was already available to us, includes beam spreading effects. While these effects are not necessary, they do not interfere with the study of crosscorrelation interpolation.

For the purposes of this paper, we simulated a 20 mm transducer focused at 120 mm. The axial impulse response at the focus is characterized by a Gaussian spectrum with 5 MHz center frequency and 60 percent fractional bandwidth ($2\sigma = 0.6 \times 5$ MHz = 3 MHz, where $\sigma$ is the standard deviation of the Gaussian). The scatterer density was adjusted to obtain an average of 48 scatterers per pulse length. Samples of the echo signals were obtained every 20 ns, corresponding to a sampling rate of 50 MHz. Delayed replicas of the echo signals were produced in software by shifting the signal by 30 samples and padding with zeros. This delay was chosen to yield enough crosscorrelation samples around the main peak to be able to test all the interpolation methods investigated in this paper. In addition to the integral sample delay, a subsample delay was applied by using the shifting property of the Fourier transform. For a subsample delay $|\delta|<0.5$ sample, the delayed signal is

$$x(n+\delta) = IFT[X(\omega_s) e^{-j\omega_s \delta}]$$

(7)

where $IFT[\cdot]$ indicates the inverse Fourier transform, $X(\omega_s)$ is the Fourier transform of the echo signal $x(n)$, and $\omega_s$ is the discrete angular frequency. Subsample delays ranging from 0.1 to 0.5 samples were used. In some cases, white noise was added to the echo signals; note that in a noiseless environment (SNR = $\infty$), Eq. (2) predicts exact time-delay estimation (zero standard deviation).

The signal characteristics were kept constant for all simulations; only the fractional part of the delays were change. Each simulation study was conducted on a set of 15 independent echo signals, 55 mm in length, centered around the focus of the transducer; using nonoverlapping, 5 mm (6.7 $\mu$s, assuming a speed of sound of 1500 m/s) rectangular data windows, we obtained 165 independent crosscorrelation time-delay estimates. The mean and standard deviation of the estimates were computed.

B. Theoretical studies

Theoretical studies were conducted assuming a cosinusoidal ccf with a Gaussian envelope. This ccf model corresponds to a Gaussian power spectrum with center frequency $f_0 = 5$ MHz, and 60 percent fractional bandwidth ($2\sigma = 3$ MHz). The ccf is given by

$$R(\tau) = e^{-\left(\frac{\tau}{\sigma}\right)^2} \cos(2\pi f_0 \tau)$$

(8)
Samples of this ccf are used to test the theoretical performance of the different curve-fitting methods.

III. PEAK INTERPOLATION TECHNIQUES

Peak interpolation techniques are used to estimate the location of the peak of the ccf from its samples. Time delays that are not integral multiples of the sampling period result in an offset between the location of the largest sample of the ccf and the true location of the peak, thus the need for interpolation. Two kinds of interpolation methods are considered:
1. Curve-fitting methods, which fit a predetermined type of curve to a number (three or more) of samples around the ccf peak; and
2. Reconstructive interpolation methods, which generate new samples of the ccf based on the original sampled ccf.

1. Curve-fitting methods

Given a sampled ccf, the location of its peak can be estimated by first fitting a curve to the samples (usually a curve that is analytically well defined), and then computing the location of the peak of the fitted curve. The time interval from the location of the largest sample of the ccf to the location of the peak of the (unsampled) ccf is called the peak offset ($\delta$) (Fig. 1). A nonzero peak offset arises when there is a subsample time delay. The time interval from the location of the largest sample of the ccf to the location of the peak of the fitted curve is the estimated peak offset ($\hat{\delta}$). The bias error, defined as

$$b_\delta = E[\hat{\delta}] - \delta,$$

is an indication of the ability of the fitted curve to approximate the ccf near the peak, as illustrated in figure 1.

![Diagram](image_url)

Fig. 1 Diagram of the crosscorrelation function peak (solid line), its samples, and a fitted curve (dashed line).
We investigate two curve-fitting methods: polynomial (parabolic) interpolation, in which a parabola is fitted to the largest three samples of the ccf, and cosine interpolation, in which a cosinusoid is fitted to the largest three samples of the ccf. As we demonstrate later, curve-fitting interpolation generally yields biased time-delay estimates [5, 12].

The interpolation approximates the ccf more accurately in the neighborhood of its peak. However, the accuracy of this approximation worsens away from the peak. The curve-fitting methods described here approximate the ccf based on the three largest samples, those that are closest to the peak of the ccf. The neighborhood of the peak is defined by an interval including these three samples. Thus, its is reasonable to expect that the absolute value of the bias error will increase with larger sampling periods; furthermore, we will demonstrate that the ratio of the bias error to the sampling period (relative bias error) also increases with larger sampling periods.

Improved curve fitting is achieved when the sampling period is small enough so that the samples used for interpolation fall within the semicycle of the ccf that contains the peak (all positive samples condition). We will demonstrate later that when this condition is not met, the bias error increases abruptly. Given a sampling period $T_s$ and a ccf with center frequency $f_0$, the all-positive-samples condition is met for any peak offset if

$$\frac{T_s}{2} + T_s = \frac{3}{2} T_s = \frac{1}{4 f_s},$$

$$\frac{1}{T_s} = 6 f.$$

Note that this condition imposes a higher lower-bound for the sampling rate than that given by the Nyquist theorem.

1a. Polynomial interpolation method

This method finds an $N^{th}$ order polynomial that best fits a predefined number of samples of the crosscorrelation function. Just as two points on a curve define a unique line passing through them, three points define a unique parabola, four points define a unique cubic, and so on. In general, using $N$ points we can find an $N^{th}$ order polynomial passing through them. However, finding the peak of fitted higher order polynomials (three and above) is tedious and computationally intensive.

To a first approximation, a region near the peak of the crosscorrelation function of a band-pass signal with a rectangular spectrum can be assumed to be a second order polynomial (parabola) [6]. For this reason and due to its simplicity, parabolic interpolation is widely used [5, 13, 22, 23]. Given the largest sample ($y_i$) and its two nearest neighbors ($y_0, y_2$), the peak-offset estimate for parabolic interpolation is

$$\hat{\delta} = T_s \frac{y_0 - y_2}{2(y_0 - 2y_1 + y_2)},$$

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where $T_s$ is the sampling period, as given in Boucher and Hassab [5], Ianniello [7], Foster et al. [13], and Champagne et al. [23].

Although fast and simple to implement, parabolic interpolation is a biased estimator of time delay, as demonstrated by Boucher and Hassab [5]. The magnitude of the bias error depends on the location of the actual peak of the ccf with respect to the location of the samples. Boucher and Hassab [5] developed general expressions of the mean and variance of the time delay that is estimated using parabolic interpolation as a function of the spectral characteristics of the signal. For a low-pass signal and white noise, the bias error is minimum when a sample of the ccf coincides with the peak ($\delta=0$) or when the peak is at the mid-point in between two samples ($\delta=0.5$); the error is maximum somewhere in between (near $\delta=0.3$) [5].

We obtained a simple analytical expression of the bias error due to parabolic interpolation by assuming a cosinusoidal model for the ccf peak. The bias error due to parabolic interpolation is given by

$$b_s = \hat{\delta} - \delta = \frac{\sin(\omega T_s) \sin(\omega \delta)}{2 \cos(\omega \delta)(1 - \cos(\omega T_s))} - \delta,$$  \hspace{1cm} (13)

where $\omega$ is the angular frequency of the cosinusoid. Eq. (13) is derived in Appendix D; note that simple expressions such as that given in Eq. (13) may be difficult to obtain for other, more general ccf shapes.

The theoretical bias error given by Eq. (13) is graphed in figure 2 for $\omega = 2\pi \times 5 \times 10^6$ radian/s, and for sampling periods ranging from 40 ns to 5 ns. This figure shows that while significant increase of the bias error occurs when $T_s$ is increased from 5 ns to 20 ns, an abrupt increase occurs when $T_s$ increases from 20 ns to 40 ns due to the violation of the all-positive-samples condition.

![Fig. 2](image_url)  
Fig. 2 Relative bias error vs. subsample part of the time delay using parabolic interpolation for sampling periods 5 ns, 10 ns, 20 ns, and 40 ns (theory).
The bias error due to parabolic interpolation was also studied using simulated echo signals sampled at $T_s = 20$ ns, with known delays ranging from 30.0 to 30.5 samples. The bias error of the time-delay estimates are shown in figure 4. In all the simulation results, the bias errors for fractional delays from -0.1 to -0.5 are assumed to be antisymmetric, as theoretically demonstrated in figure 2. As predicted by the results obtained using the theoretical cosine ccf model, the simulation results show a bias error that is a function of the peak offset (the subsample part of the time delay); the bias error is maximum when the actual peak of the sampled crosscorrelation function is 0.3 samples away from the highest sample. This also confirms the results of Boucher and Hassab [5].

If the shape of the peak of the crosscorrelation function is known exactly, a time-delay bias error formula can be obtained and used to compensate (unbias) the estimates. For example, Eq. (13) could be used to correct bias errors when the ccf is a cosinusoid. However, such a compensation scheme would not reduce the variability around the bias introduced by the curve fitting algorithm (figure 4). A simplified version of this compensation technique was used by Kallel and Bertrand [24]; under the assumption that the fractional shift is smaller than one fourth of the sampling period, the bias error was linearly approximated. The slope of this line was used as a correction factor to compensate for the bias.

Unfortunately, the exact shape of the ccf is generally unknown. The ccf function varies with depth due to frequency dependent attenuation, with changes in the scattering properties of the insonified tissues, with the scatterer distribution, and with the location of the echo with respect to the ultrasonic beam. Thus, a compensation procedure based on a theoretical and/or mean shape of the ccf peak will not remove the bias completely. In section 1c below, we demonstrate the dependence of the parabolic interpolation error on the bandwidth and center frequency of the echo signal.

1b. Cosine-fit interpolation

Another curve that has been used by DeJong et al. [12, 14] to interpolate the ccf peak is the cosine function. This model for the ccf is reasonable for some signals with rectangular and Gaussian spectra (see Appendix D). These investigators used cosine-fit interpolation in an application of time-delay estimation to blood velocity estimation. Using 64 samples at a signal-to-noise ratio of 10 and a sampling rate 5 times the center frequency of the echo-signals, they reported a standard deviation of the error in the determination of time delay of 0.08 samples.

In DeJong et al. [14], the peak of a cosinusoid fitted to the three largest samples of the digital ccf $\gamma_0, \gamma_1, \gamma_2$ was derived to be

$$\hat{\delta} = -\frac{\theta}{\omega_o}$$  \hspace{1cm} (14)

where $\omega_o$ is the angular frequency of the cosinusoid given by

$$\omega_o = \cos^{-1} \left( \frac{\gamma_0 + \gamma_2}{2\gamma_1} \right),$$  \hspace{1cm} (15)
and $\theta$ is the phase of the cosinusoid given by

$$\theta = \tan^{-1}\left(\frac{y_0 - y_1}{2y_1 \sin \omega}\right).$$

(16)

Theoretical values of the bias error obtained from Eq. (16) are shown in figure 3 for sampling periods ranging from 5 ns to 40 ns. This figure shows that while significant deterioration occurs when $T_s$ is increased from 5 ns to 20 ns, an abrupt increase occurs when the $T_s$ increases from 20 ns to 40 ns due to the violation of the all-positive-samples condition.

The bias error obtained from the simulated echo-signals is shown in figure 4. As in the parabolic interpolation case, the simulation results closely approximate the theoretical results, with the bias error being maximum near $\delta = 0.3$, and minimum at $\delta = 0$ and $\delta = 0.5$.

1c. Sensitivity and comparison of cosine and parabolic interpolation

The bias error of the curve-fitting methods is a result of the inability of the fitted curve to match the shape of the ccf exactly. It is logical to expect that for a given fitting curve (e.g., parabola), different bias errors will occur for different shapes of the ccf. This is illustrated in figure 5, which shows theoretical bias errors obtained by fitting a parabola to two different ccf models: a cosinusoidal ccf and a Gaussian ccf. For both models, the bias error reaches a maximum when the peak is approximately 0.3 $T_s$ from the largest sample ($\delta = 0.3$). The bias error is smaller when the parabola is fitted to a cosine ccf peak than when it is fitted to the Gaussian envelope ccf (Fig. 5). This demonstrates the dependence of the parabolic interpolation error on the shape of the ccf.

![Bias error vs. time delay](image)

**Fig. 3** Relative bias error vs. subsample part of the time delay using cosine interpolation for sampling periods 5 ns, 10 ns, 20 ns, and 40 ns (theory).
The bias errors obtained using parabolic and cosine interpolations are shown in figures 2 and 3; we can observe that the bias error due to the cosine-fit interpolation is generally smaller than that due to the parabolic interpolation. This is true for the Gaussian envelope ccf described earlier; different results can be expected for different shapes of the ccf.

Using the Gaussian envelope ccf, we performed a comparative study of the dependence of the bias error on bandwidth and center frequency using the cosine and parabolic interpolation methods. In both cases, the bias error increases with
increased bandwidth (figures 6, 7). The dependence of the cosine interpolation on bandwidth was noted previously by DeJong et al. [12]. Comparing figures 6 and 7, we observe that even though cosine interpolation is more sensitive to bandwidth changes than parabolic interpolation, it is also significantly more accurate. The bias error increases with increased center frequency (figures 8, 9). Both approaches have similar sensitivity to center frequency changes.

Curve-fitting interpolation has been shown to produce biased estimates of time delay, with a bias error that is dependent on the fractional sample part of the time delay. Furthermore, the bias error is also a function of the spectral characteristics of the signals and the sampling period. Thus the need to consider reconstructive interpolation.

2. Reconstructive interpolation

The sampling theorem states that samples of a continuous-time bandlimited signal sampled at or above the Nyquist rate are sufficient to represent the signal exactly in the sense that the signal can be recovered from the samples. Based on this principle the reconstructive interpolation methods recover samples of the ccf to improve the precision of time delay estimation.

2a. Zero-padding method

The crosscorrelation function can be computed in the time domain by direct implementation of Eq. (6) or in the Fourier domain. In the Fourier domain, the crosscorrelation function can be computed as

\[ \hat{R}_{xy}(\tau) = IFT\{X(\omega_k) \times Y^*(\omega_k)\}, \]  

(17)

where \( X(\omega_k) \) and \( Y(\omega_k) \) are the discrete-time Fourier transforms of the echo-signal segments and \( Y^*(\omega_k) \) is the complex conjugate of \( Y(\omega_k) \). When the ccf function is

![Graph showing relative bias error of the parabolic interpolation vs. bandwidth expressed as the std. dev. of the center frequency (Ts = 20 ns).](image)

Fig. 6 Relative bias error of the parabolic interpolation vs. bandwidth expressed as the std. dev. of the center frequency (Ts = 20 ns).
Fig. 7 Relative bias error of the cosine interpolation vs. bandwidth expressed as the std. dev. of the center frequency (Ts = 20 ns).

used for time-delay estimation, the time-domain and Fourier-domain methods perform similarly [10]. However, in applications where the approximate value of time delay is known a priori, the time-domain method can result in significant computational savings since fewer lags of the ccf need to be computed [10].

When the ccf is computed using the Fourier domain approach, new samples of the ccf can be obtained by appending zeros to the cross-spectral product in Eq. (17) before the inverse Fourier transformation [25]. This operation results in a decrease of the sample spacing commensurate with the number of appended zeros. For

Fig. 8 Relative bias error of the parabolic interpolation vs. center frequency (Ts = 20 ns).
Fig. 9  Relative bias error of the cosine interpolation vs. center frequency (Ts = 20 ns).

example, to double the effective sampling rate, we must double the length of the complex cross-spectral product by appending zeros. Therefore, this procedure can be computationally intensive when the desired time-delay resolution is much smaller than the sampling period. More importantly, the zero-padding method is highly inefficient since it provides interpolated samples for the entire crosscorrelation function, whereas for the purpose of TDE, we only need dense interpolation near the area of the peak. An approach that is theoretically equivalent to the zero-padding method is reconstructive interpolation.

2b. Reconstruction interpolation

This method is a time-domain technique based on analog signal reconstruction theory. The theory of signal reconstruction is described by Oppenheim and Schafer [26, Chapter 3] and is not detailed here. It is possible to obtain additional samples of the crosscorrelation function at arbitrary times based on the samples (obtained at or above the Nyquist sampling rate) by means of a low-pass filter. The reconstruction filter approach was used by Stein [22] to reconstruct samples of the crossambiguity function using a raised-cosine filter.

The impulse response of the ideal low-pass filter is the sinc function that it is not realizable in practice since it is infinitely long. In this paper, we use an FIR implementation of the low-pass filter that consists of windowing the ideal low-pass filter with a Blackman tapering window [26]. Because the interpolated samples are ideally obtained using a filter with a sinc-shaped impulse response, this procedure is also known as sinc interpolation (not to be confused with the interpolation method consisting of fitting a sinc function to the peak of the crosscorrelation function). The interpolated samples are obtained using

$$x(t) = \sum_{n=-N/2}^{N/2} x(n) \text{sinc} \left[ \pi \left( t - n Ts \right) / Ts \right] w \left( t - n Ts \right) / Ts$$

(18)
where $T_s$ is the sampling period, $N_f$ is the length of the filter, and $w_k()$ is the Blackman window [26]. The time $t$ assumes values in between the times corresponding to the known samples. To guarantee that there are $N_f/2$ crosscorrelation samples to the left of the peak, we provide an arbitrary initial shift of $N_f/2$ samples to the shifted echo signal.

Inaccuracies of the reconstructive interpolation method were studied using simulated echo signals with known subsample delays ranging from 0 to 0.5 samples. The mean of the time-delay estimates are shown in figure 4; these results demonstrate that reconstructive interpolation yields an unbiased estimator of time delay.

As a result of the finite length of the reconstruction filter, finite resolution calculations and noise, the improvement in time resolution achieved by reconstructive interpolation is limited. We hypothesize that, in a noisy environment, there is a maximum number of interpolated samples above which the improvement in time resolution is overcome by the errors in the interpolation. In order to test this hypothesis, we have investigated empirically the improvement of the time-delay error with increased number of interpolated samples using simulated echo signals with and without additive white noise. The number of interpolated samples was increased from 1 to 500; the standard deviation of the time delay are shown in figure 10a. Note that since the time delay in this test was chosen to have a subsample part equal to 0.5 samples, two different results are obtained for odd and even number of interpolated samples. In the noiseless case, for odd number of interpolated samples the error is zero since an interpolated sample is always placed at the exact location of the peak as illustrated in figure 11. For an even number of interpolates, the ccf peak is always $\pm T_s/2/M$ sampling periods away from the nearest sample, where $M$ is the number of interpolated samples (Fig. 11). Thus, the standard deviation of the time-delay estimate is $T_s/2/M$. This result implies that the standard deviation would improve indefinitely with increasing number of interpolates, approaching zero for infinite $M$. However, when these signals contain additive noise, the standard deviation approaches the CRLB for both odd and even number of interpolates (Fig. 10b). Note that at about 160 interpolates, the odd interpolation density is sufficient to detect the small errors due to the presence of noise and therefore there is a step change in precision. Therefore, when noise is present, a finite number of interpolated samples is required to achieve the maximum precision; in our -40 dB white noise example this occurs at about 350 interpolates. The results of figure 10 are for a subsample shift of 0.5 samples; similar behavior but of different magnitude is expected for other subsample shifts.

We also have investigated empirically the effect of filter length. Truncated sinc-shaped impulse responses ranging from 6 to 35 samples were used to estimate a 30.5 sample time delay on simulated echo-signals. The results are shown in figure 12. In the noiseless case, the time-delay error remains constant for filter lengths larger than 16 samples (at 50 MHz sampling rate). When the -40 dB white noise is added, the error approaches the CRLB for filter lengths >12 samples. Although the trend in figure 12 can be expected to be generally applicable, some variations may be obtained for other filter implementations and/or signal spectral characteristics. Optimum interpolation filters are discussed in detail by Schafer et al. [27].
Fig. 10 Standard deviation of the reconstructive interpolation error vs. number of interpolated samples ($Ts = 20$ ns) (filter length, $Nf = 35$): a) noiseless case, and b) -40 dB noise.

2c. Iterative reconstructive filtering

Reconstructive interpolation can be computationally intensive. A faster alternative is to perform interpolation filtering in an iterative way [22]. The procedure is illustrated in figure 13. First, we reconstruct a sample in between the two highest original samples of the ccf. The value of the reconstructed sample will be higher than or equal to the value of (at least) one of the original samples. Then, we reconstruct a new sample in between the last interpolate and the highest of the two original samples. We find the highest two of these three and the process is repeated. In figure 13, the reconstructed samples (crosses) are tagged with the corresponding iteration number in which they were originated. In a similar application, the iterative approach was used by Stein [22] to find the value of the crossambiguity function at an arbitrary time.
Using the iterative method, we achieve a subsample resolution of $2^N$, where $N$ is the number of iterations, i.e., a $2^N/N$ savings in computation. For example, an interpolation level of 250 samples can be achieved in 8 interpolation steps, a $256/8 = 32$-fold interpolation time savings, resulting in comparable processing time as that obtained using parabolic interpolation. Using the iterative approach we obtain the same TDE precision and accuracy as the noniterative approach at a fraction of the computational expense. For this reason, independent simulation results are not separately obtained for the iterative method.

Fig. 12 Standard deviation of the reconstructive interpolation error vs. filter length ($T_s = 20$ ns; data length 5 mm; 350 interpolated samples). The 95 percent confidence interval for the standard deviation estimates is approximately ±10 percent (N=165).
IV. EFFECT OF BIAS ERRORS ON PRECISION

The expression for the CRLB given in Eq.(1) is valid exclusively for unbiased estimators; therefore, it is not applicable to the precision of time delay estimates obtained using curve-fitting interpolation methods. For biased estimators the CRLB is given by

$$\sigma_{CRLB}^b = \sigma_{CRLB} \left(1 + \frac{d b_b(\delta)}{d \delta} \right)$$  \hspace{1cm} (19)

where the bias error $b_b$ is expressed as a function of the fractional sample delay $\delta$ [28, 29]. Note that since the bias error is a function of $\delta$, the CRLB of biased estimators is also a function of $\delta$.

To evaluate the magnitude of the increase of the CRLB due to bias errors, we approximate the derivative in Eq.(19) by a finite difference as follows,

$$\sigma_{CRLB}^b \cong \sigma_{CRLB} \left(1 + \frac{b_b(\delta_2) - b_b(\delta_1)}{\delta_2 - \delta_1} \right),$$  \hspace{1cm} (20)

where $\delta_1$ and $\delta_2$ are two fractional time delays. From all the results discussed in this paper, the largest change in bias error occurs for parabolic interpolation on signals sampled at 25 MHz (Fig. 2), near $\delta = \pm 0.5$; plugging these values into Eq.(20), we obtain

$$\sigma_{CRLB}^b \cong \sigma_{CRLB} \left(1 + \frac{(0.026 - 0)}{(-0.4 - (-0.5))} \right) = 1.26 \sigma_{CRLB}.$$  \hspace{1cm} (21)

On the other hand, the smallest change in bias error occurs near $\delta = \pm 0.3$ where the CRLB remains practically unaltered.
Thus, the effect of bias errors in time-delay estimation appears as a deterioration of the precision; the magnitude of this deterioration is a function of the fractional part of the time delay. Since the CRLB is related to the gradient of bias error with $\delta$ and the bias error is periodic with $\delta$, bias and precision error are out of phase (the bias is maximum when the precision is minimum and vice versa). However, the magnitude of the increase of the CRLB appears to be small ($\leq 26$ percent) for all the cases included in our study.

V. SUMMARY AND CONCLUSIONS

The theoretically-achievable precision of any time-delay crosscorrelation estimator is given by the Cramér-Rao lower bound. When using digital crosscorrelation under high signal-to-noise ratio conditions, the TDE error due to time quantization can be large compared to the Cramér-Rao lower bound. The reason is that the location of the largest sample of the discrete ccf can be as far as one half the sampling period apart from the peak of the ccf. Therefore, applications of TDE that require high precision must involve techniques that interpolate the ccf peak. For example, such is the case in elastography and in some blood flow measurements, where the time resolution available from discrete-time crosscorrelation TDE is insufficient and precludes the generation of meaningful images. The limited time resolution of the digital crosscorrelation is the motivation for investigating the methods for interpolation of the discrete-time crosscorrelation. We have classified these into two kinds of methods: curve-fitting and reconstructive methods.

Two curve-fitting methods that have been used in the past by other investigators were evaluated and compared. In general, time-delay estimates obtained by curve-fitting methods are biased such that the bias error is a function of the subsample part of the time delay. We have demonstrated this effect theoretically and validated the results with simulations, which match the theoretical predictions well.

The bias error introduced by curve-fitting interpolation is a function of the appropriateness of the fitted curve; in our tests on typical ultrasound signals, cosine interpolation performed better than parabolic interpolation. While a parabola or a cosine curve may fit a ccf well in an arbitrarily small time interval near the peak, the goodness of the fit deteriorates with increased length of this interval. When using three-point interpolation, the length of the interval is two sampling periods. We have shown that the sampling period can strongly affect the bias error. This is an important consideration when comparing results obtained by different investigators. In practice, the dependence of the bias error on the sampling period imposes a higher lower-bound for the sampling rate than that given by the Nyquist theorem. A practical consideration to avoid very large bias errors is to have the sampling rate exceed six times the ccf center frequency.

We have demonstrated that the bias error of the curve-fitting interpolation methods is also a function of the spectral characteristics of the signals. Therefore, in medical ultrasound applications, bias compensation based on mean bias estimates may be inaccurate since the spectral characteristics of the echo signal are spatially variant. The main advantage of curve-fitting interpolation is its computational
simplicity; this makes it the method of choice in applications where the bias error that results from suboptimal interpolation is acceptable.

An additional detrimental effect of the bias error is that it increases the CRLB. However, we have shown that this increase is small even for parabolic interpolation and low sampling rates. This suggests that the CRLB derived for unbiased estimators may still be useful in most practical cases.

Significant deleterious effects of the bias error may occur in applications that track a continuously-varying time delay as a function of a parameter such as time or depth. In this case, the periodic nature of the bias error (one cycle of which is shown for example in figure 2) results in ripple in the estimated time delay function. For example, a particular image artifact due to the time-delay bias errors occurs in elasticity imaging using elastography; elastograms (elasticity images) are corrupted by the "zebra line" artifact that appears as a sequence of periodic bright and dark lines [10]. Concurrent with local changes in strain and echo-signal spectra, the frequency, intensity, and spatial distribution of the artifact changes, and therefore it is hard to filter by conventional means. The artifact can be avoided by using reconstructive interpolation. To our knowledge, the detrimental effect of bias errors have not been reported in blood flow or other tissue motion applications that involve TDE.

Reconstructive interpolation can minimize TDE bias errors at the expense of added computational cost. Unlike curve-fitting interpolation, the performance of reconstructive interpolation does not depend on the shape of the ccf, as long as the echo signals have been appropriately sampled (sampling rate above the Nyquist rate). Thus, reconstructive interpolation is an unbiased estimator of the location of the ccf peak. We have shown that for typical medical ultrasound signals (5 MHz center frequency, 60% fractional bandwidth, and 50 MHz sampling rate) and -40 dB additive white noise, reconstructive interpolation achieves the CRLB with 350 interpolates and a 12 point reconstruction filter, rendering inefficient any further sophistication of the reconstructive interpolation procedure. In order to reduce the computational cost of reconstructive filtering, an efficient iterative approach was described, which results in a $2^N/N$ reduction in computation, where $N$ is the number of iterations. This reduction in computation is achieved without any deterioration of the bias and precision of the time-delay estimates.

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APPENDIX A

The Cramér-Rao bound on the precision of time delay estimation.

The focus of this paper is time delay estimation under high signal-to-noise ratio conditions. However, for completeness and to allow the comparison with other
Case 1. Rectangular spectrum

The integrals in Eqs. (A6) and (A7) can be easily computed for the case of a rectangular spectrum. Assuming a signal and noise spectra which are constant over a band extending from \( f_1 \) to \( f_2 \) (and from \(-f_1\) to \(-f_2\)), with spectral density power \( S_o/2 \) and \( N_o/2 \), respectively, from Eq.(A7) we obtain

\[
\sigma_{CR}^2 \equiv \left\{ \frac{2T4\pi^2 S_o}{2N_o} \int_{f_1}^{f_2} f^2 df \right\}^{-1} = \left\{ \frac{4\pi^2 T S_o}{N_o} \left( \frac{f_2^3 - f_1^3}{3} \right) \right\}^{-1}.
\]  
(A8)

This expression coincides with Eq. (23) in Quazi [2].

Using the substitutions

\[
f_2 = f_o - B/2 \quad \text{and} \quad f_1 = f_o + B/2,
\]  
(A9)

and using the definitions of signal and noise energy and signal-to-noise ratio given by

\[
S = S_o/2(f_2 - f_1) + S_o/2(-f_1 + f_2) = S_o(f_2 - f_1),
\]  
(A10)

\[
N = N_o/2(f_2 - f_1) + N_o/2(-f_1 + f_2) = N_o(f_2 - f_1),
\]  
(A11)

\[
\text{SNR} = S/N = S_o/N_o,
\]  
(A12)

we obtain after some algebra,

\[
\sigma_{CR}^2 \equiv \frac{1}{4\pi^2} \frac{1}{f_o^2} \frac{1}{B} \frac{1}{T} \frac{1}{\text{SNR}} \left( 1 + \frac{B^2}{12f_o^2} \right).
\]  
(A13)

When \( B^2 / 12f_o^2 \ll 1 \), the approximate CR bound for high SNR signals is

\[
\sigma_{CR}^2 \equiv \frac{1}{4\pi^2} \frac{1}{f_o^2} \frac{1}{B} \frac{1}{T} \frac{1}{\text{SNR}}.
\]  
(A14)

which coincides with the expression derived by Weinstein and Weiss [30], after reconciliation of the definitions of signal-to-noise ratio.

A similar development using Eq. (A7), which can be found in Quazi [2, Eq.(22)], yields the CR bound for low SNR signals,

\[
\sigma_{CR}^2 \equiv \frac{1}{8\pi^2} \frac{1}{f_o^2} \frac{1}{B} \frac{1}{T} \frac{1}{\text{SNR}^2} \left( 1 + \frac{B^2}{12f_o^2} \right).
\]  
(A15)

When \( B^2 / 12f_o^2 \ll 1 \) (relative bandwidth = \( B_o / f_o \ll 3.46 \)), the approximate CR bound for high SNR signals is

\[
\sigma_{CR}^2 \approx \frac{1}{8\pi^2} \frac{1}{f_o^2} \frac{1}{B} \frac{1}{T} \frac{1}{\text{SNR}^3}.
\]  
(A16)
Case 2. Gaussian spectrum

In medical ultrasound studies, it is common to assume a Gaussian-shaped spectrum, rather than a rectangular spectrum. We derive the CR bound for Gaussian spectra.

Assume a noise spectrum which is constant with spectral density power \( N_o / 2 \), and a signal Gaussian signal spectrum with peak spectral power \( S_o / 2 \) given by

\[
S(f) = \frac{(f-f_o)^2}{2\sigma^2} + \frac{(f+f_o)^2}{2\sigma^2}, \quad -\infty < f < \infty
\]

(A17)

where \( f_o \) is the center frequency and \( \sigma \) is the standard deviation. Since the spectrum is not flat, the CR variance cannot be derived using Eqs. (A6) or (A7), which are valid for low or high signal-to-noise ratio conditions across the bandwidth. However, an approximate CR bound may be obtained by defining the signal-to-noise ratio of the equivalent noise bandwidth spectrum.

In theory, given an infinite-bandwidth Gaussian spectrum and white noise, the definition of signal-to-noise ratio results in \( \text{SNR} = 0 \), a result of little practical utility. Thus, for the purposes of obtaining an expression of the CR bound, it is useful to replace the Gaussian for a substitute bandlimited spectrum. One approach is to use a rectangular spectrum, centered at the same frequency as the Gaussian and which produces a signal with the same mean square value. The bandwidth of such a spectrum is the equivalent noise bandwidth defined in Appendix B. Substituting \( B_n = \sqrt{2\pi} \sigma \), the equivalent noise bandwidth signal-to-noise ratio is \( \text{SNR}_n = S_o B_n / N_o B_n = S_o / N_o \). Then, assuming a high \( \text{SNR}_n \), we use Eq. (A6) and Eq. (A11) to obtain the approximate CR minimum variance,

\[
\sigma_{\text{CR}}^2 \equiv \left\{ 2T \int_{-\infty}^{\infty} \frac{(f-f_o)^2}{2\sigma^2} \frac{S_o}{2N_o} e^{-\frac{(f-f_o)^2}{2\sigma^2}} df \right\}^{-1} = \left\{ 4\pi^2 T \frac{S_o}{N_o} \int_{-\infty}^{\infty} f^2 e^{-\frac{(f-f_o)^2}{2\sigma^2}} df \right\}^{-1}
\]

(A18)

With the change of variables, \( u = \frac{(f-f_o)}{\sqrt{2} \sigma} \), the integral in Eq. (A18) can be expressed as

\[
I = \int_{-\infty}^{\infty} f^2 e^{-\frac{(f-f_o)^2}{2\sigma^2}} df
\]

(A19)

\[
I = 2\sqrt{2}\sigma^3 \int_{-\infty}^{\infty} u^2 e^{-u^2} du + \sqrt{2}\sigma f_o^2 \int_{-\infty}^{\infty} u e^{-u^2} du + 4\sigma^2 f_o^2 \int_{-\infty}^{\infty} e^{-u^2} du
\]

(A20)

\[
I = 2\sqrt{2}\sigma^3 \sqrt{\pi} / 2 + \sqrt{2}\sigma f_o^2 \sigma \sqrt{\pi} + 0
\]

(A21)

\[
I = \sqrt{2\pi} \sigma f_o^2 \left( 1 + \frac{\sigma^2}{f_o^2} \right)
\]

(A22)
and we obtain the CR bound for Gaussian spectrum in white noise,
\[
\sigma_{CR}^2 \approx \frac{1}{4\pi^2 \sqrt{2\pi}} \frac{1}{T \sigma} \frac{S_0}{N_0} \frac{1}{f_o^2} \left(1 + \sigma^2/f_o^2\right).
\]  
(A23)

Substituting \( \text{SNR}_n = S_0 B_n/N_0 B_n = S_0/N_0 \) in Eq. (23), we get
\[
\sigma_{CR}^2 \approx \frac{1}{4\pi^2} \frac{1}{TB_n \text{SNR}_n f_o^2} \left(1 + B_n^2/2\pi f_o^2\right).
\]  
(A24)

In the case of \( B_n^2/2\pi f_o^2 << 1 \) (relative bandwidth \( = B_n/f_o << 2.5 \)), Eq. (A24) can be approximated by
\[
\sigma_{CR}^2 \approx \frac{1}{4\pi^2} \frac{1}{f_o^2} \frac{1}{TB_n \text{SNR}}.
\]  
(A25)

The above expression is the same as that obtained for a rectangular spectrum of bandwidth \( B = B_n \). The validity of the use of the equivalent noise bandwidth spectra to obtain approximate CR bound expressions can be good for many spectral shapes [4].

APPENDIX B

Derivation of the equivalent noise bandwidth for a Gaussian power spectrum.

The equivalent noise bandwidth is defined by
\[
B_n = \frac{\int_{-\infty}^{\infty} S(f) \, df}{S(f)_{\text{max}}}.
\]  
(B1)

[6, p.141], as the bandwidth of a rectangular spectrum with the same power and peak amplitude as the actual signal spectrum, \( S(f) \).

Given an ultrasound signal with Gaussian two-sided power spectrum
\[
S(f) = \frac{(f-f_c)^2}{2\sigma^2} + \frac{(f+f_c)^2}{2\sigma^2}, \quad -\infty < f < \infty
\]  
(B2)

where \( f_c \) is the center frequency and \( \sigma \) is the standard deviation (for most practical values of \( f_c/\sigma \), the DC component of this signal model is negligible), we obtain
\[
\int_{-\infty}^{\infty} S(f) \, df = S_o \int_{-\infty}^{\infty} e^{-\frac{(f-f_c)^2}{2\sigma^2}} \, df = S_o \sqrt{2\pi} \sigma
\]  
(B3)

\[
S(f)_{\text{max}} = S_o.
\]  
(B4)
Therefore, the equivalent noise spectral bandwidth of a Gaussian spectrum is

\[ B_n = \sqrt{2\pi} \sigma \approx 2.5 \sigma. \]  

(B5)

A similar derivation can be found in [6, p.142] for a low-pass Gaussian spectrum.

APPENDIX C

We evaluate the error of the cosine fit to the peak of the crosscorrelation function originated from signals with uniform and Gaussian spectra. In the context of three-point curve fitting, we are interested in the validity of the approximation as far as \(3/2\) sampling periods away from the peak.

**Signals with uniform spectrum**

Given a signal pair \(x(t)\) and \(y(t)\) with a uniform bandpass spectrum with center frequency \(f_o\) and bandwidth \(B\), the crosscorrelation function between these two signals is given by

\[ R_{xy}(\tau) = B \left( \frac{\sin \pi B \tau}{\pi B \tau} \right) \cos(2\pi f_o \tau) \]  

(C1)

For typical values of \(f_o = 5\) MHz, \(B = 3.5\) MHz (70 percent fractional bandwidth) and \(T_s = 20\) ns (sampling rate 50 MHz), the term

\[ \left( \frac{\sin \pi B \tau}{\pi B \tau} \right)^{\tau=3T_s/2} \approx 0.9819 \approx 1, \]  

(C2)

and the term

\[ \cos(2\pi f_o \tau)^{\tau=3T_s/2} \approx 0.5878. \]  

(C3)

Since in the region of the peak the sinc term is much closer to one than the cosine term, the crosscorrelation can be approximated by

\[ R_{xy}(\tau) \approx \cos(2\pi f_o \tau), \]  

(C4)

for

\[ \frac{-3T_s}{2} \leq \tau \leq \frac{3T_s}{2}. \]  

(C5)

**Signals with Gaussian spectra**

Given a signal pair \(x(t)\) and \(y(t)\) each with Gaussian spectrum

\[ S_x(f) = S_y(f) = e^{-\frac{f^2}{2\sigma^2}} \otimes [\delta(f+f_o) + \delta(f-f_o)], \]  

(C6)
where \( \delta(t) \) is the unit impulse function, their cross-spectrum is given by

\[
S_{xy}(f) = e^{-\frac{f^2}{2\sigma^2}} \otimes \left[ \delta(t) \cos(f_0 t) + \delta(t - f_0 t) \right],
\]

(C7)

where \( \otimes \) indicates convolution. The corresponding crosscorrelation function is given by the inverse Fourier transform of the cross-spectrum,

\[
R_{xy}(\tau) = e^{-\frac{\pi^2 r^2}{2}} \left( \frac{\cos 2\pi f_0 \tau}{2} \right).
\]

(C8)

Note that the crosscorrelation function is formed by a Gaussian envelope decay and a cosinusoidal term. Using the same parameters as in the previous example (\( f_0 = 5 \) MHz, \( B = 3.5 \) MHz, \( T_s = 20 \) ns), the values of the envelope and cosinusoidal terms for the maximum distance between the peak of the crosscorrelation and a sample are,

\[
e^{-\frac{\pi^2 r^2}{2}} \left( \frac{\cos 2\pi f_0 \tau}{2} \right) \equiv 0.9657
\]

(C9)

\[
\cos 2\pi f_0 \tau \left( \frac{\cos 2\pi f_0 \tau}{2} \right) \equiv 0.5878.
\]

(C10)

Comparing this result with the previous example, we can observe that the cosinusoidal model is a worse approximation to the shape of the crosscorrelation within \( \pm 3 T_s / 2 \) of the peak for a Gaussian-envelope crosscorrelation.

**APPENDIX D**

Here we describe the bias error incurred when measuring time-delay by fitting a parabola to the three largest samples of a cosinusoidal ccf.

Given the three largest samples, \( (y_0, y_1, y_2) \), of the crosscorrelation function \( R(t) \),

\[
y_0 = R(t_0), \quad y_1 = R(t_1), \quad y_2 = R(t_2),
\]

(D1)

the estimated peak-offset (distance from the largest sample to the actual peak of the function) for the parabola that passes through those three samples is

\[
\hat{\delta} = \frac{1}{T_s} \frac{y_0 - y_2}{2 y_2 - 2 y_1 + y_0},
\]

(D2)

where \( \hat{\delta} \) is an estimate of the peak offset \( \delta \), and \( T_s \) is the sampling period [7, 15, 26].

If we assume a cosinusoidal model for the ccf peak, we get

\[
R(t) = A \cos(\omega t + \delta),
\]

(D3)
and the three largest samples of this ccf are given by,

$$ y_i = R(t_i) = A \cos(\omega t_i - \delta) = A(\cos \omega t_i, \cos \omega \delta - \sin \omega t_i, \sin \omega \delta), \quad (D4) $$

with $i=1, 2, \text{ and } 3$.

Without loss of generality we can assume that $t_1 = 0$, and therefore

$$ t_2 = t_1 - T_s = -T_s, \quad \text{and} \quad (D5) $$

$$ t_3 = t_1 + T_s = T_s. \quad (D6) $$

Plugging these into Eq. (D4), we get

$$ y_0 - y_2 = 2A \sin \omega T_s \sin \omega \delta \quad (D7) $$

$$ y_0 + y_2 = 2A \cos \omega T_s \cos \omega \delta \quad (D8) $$

$$ y_1 = A \cos(-\omega T_s). \quad (D9) $$

Plugging these into Eq. (D2), we get

$$ \frac{\hat{\delta}}{T_s} = \frac{1}{2} \frac{\sin \omega T_s \sin \omega \delta}{1 - \cos \omega T_s \cos \omega \delta} = \frac{1}{2} \frac{\sin \omega T_s \sin \omega \delta}{2 \cos \omega \delta (1 - \cos \omega T_s)}. \quad (D10) $$

For

$$ \omega T_s \to 0 \quad (D11) $$

$$ \omega \delta \to 0 $$

we get,

$$ \cos \omega \delta \to 1 \quad (D12) $$

$$ \sin \omega T_s \to \omega T_s $$

$$ \sin \omega \delta \to \omega \delta $$

and using a series expansion we get,

$$ (1 - \cos \omega T_s) \to \frac{(\omega T_s)^2}{2!} \quad (D13) $$

and, therefore,

$$ \hat{\delta} \to \delta \quad (D14) $$

which means that if the sampling rate is sufficiently high with respect to the frequency of the crosscorrelation, the parabolic interpolation would give unbiased estimates.

Eq. (D10) can be used as a correction formula to eliminate the bias introduced by the parabolic assumption. This correction is exact only in the case of a cosinusoidal crosscorrelation peak. Otherwise, Eq. (D10) does not completely remove the bias error.
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