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The Combined Effect of Signal Decorrelation and Random Noise on the Variance of Time Delay Estimation

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Abstract—Tissue motion and elasticity imaging techniques commonly use time delay estimation (TDE) for the assessment of tissue displacement. The performance of these techniques is limited because the signals are corrupted by various factors including electronic noise, quantization, and speckle decorrelation. Speckle decorrelation is caused by changes in the coherent interference among scatterers as the tissue moves relative to the ultrasound beam.

In time delay estimation, the effect of noise is usually addressed through the signal-to-noise ratio (SNR) term. Decorrelation, often a significant source of error in medical ultrasound, is commonly described in terms of the correlation coefficient. A relationship between the correlation coefficient and the SNR was previously derived in the literature [1], for identical signals corrupted by uncorrelated random noise. In this paper, we derive the relationship between the peak of the correlation coefficient function and the SNR for two jointly stationary signals when a delay is present between the signals. Recently, an expression for the Cramér-Rao lower bound (CRLB) has been derived in the literature for partially decorrelated signals in terms of the SNR and the correlation coefficient [2]. Since the applicability of the CRLB is determined not only by the SNR, but also by the correlation coefficient, it is important to unify the expression for the CRLB for partially correlated signals. In this paper, we derive an expression for the CRLB in terms of an equivalent SNR converted from the correlation coefficient using an SNR-$\rho$ relationship, and show this expression to be equivalent to the expression for CRLB in [2].

We also corroborate the validity of the SNR-$\rho$ expression with a simulation. Using this formulation, correlation measurements can be converted to SNR to obtain a composite SNR. The use of this composite SNR in lieu of those in the CRLB expression in the literature allows the extension of the literature results to the solution of the common TDE problems that involve signal decorrelation.

I. INTRODUCTION

Tissue motion and elasticity imaging techniques commonly use time delay estimation (TDE) for the assessment of tissue displacement [3]–[7]. The performance of these techniques is tied to the performance of the time delay (or displacement) estimation algorithms. In medical ultrasound, time delay estimation is rarely exact because it is performed between a reference signal and a delayed replica of the signal and these are corrupted by various elements including electronic noise, quantization, and decorrelation. Decorrelation can be caused by relative translation of the beam and the tissue during the time interval between the acquisition of the reference and the delayed signals.

The theory that describes the achievable precision in TDE is commonly expressed in terms of the signal and noise properties such as bandwidth, center frequency, data length, and signal-to-noise ratio (SNR). For example, at a relatively high SNR, the minimum variance of a time delay estimator is given by the Cramér-Rao lower bound (CRLB) [8]–[10]. Virtually all the literature concerning TDE discusses the CRLB in terms of SNR only [11]–[19]. Unfortunately, decorrelation, often a significant source of error in medical ultrasound, has not been incorporated in these analyses until recently, when Walker and Trahey derived an equation for the CRLB incorporating the correlation coefficient, thereby allowing the inclusion of the effect of decorrelation [2]. The CRLB is only applicable to high SNR situations; at lower SNRs, the Barankin bound or the Ziv-Zakai bound should be applied instead [16]. The thresholds of applicability of these bounds are described in terms of the post-integration SNR only, which is defined as the product of bandwidth, data length, and SNR [16] (and not in terms of correlation coefficient function). However, if even a slight decorrelation is present, the CRLB may not be a realistic lower bound even at high SNR, and thus the thresholds need to be modified accordingly. As a result, with the knowledge of decorrelation and SNR as disjoint entities, the CRLB expression in [2] might be inappropriately applied.

In this paper, we have formulated a signal model that allows us to obtain an expression of the CRLB which includes an additional SNR term for decorrelation. We have also derived the relationship between the peak of correlation coefficient function and the SNR for two jointly stationary signals. Using the SNR-$\rho$ relationship, our expression for the CRLB is shown to be equivalent to that derived in [2]. The validity of the SNR-$\rho$ relationship is corroborated by simulation. Using this relationship, correlation measurements can be unified with SNR, which will allow us to use the expressions available throughout the TDE literature that are in terms of SNR only.

II. CORRELATION COEFFICIENT FUNCTION AND SNR

The expression for Cramér-Rao lower bound for partially decorrelated speckle derived by Walker and Trahey [2] is as follows:

$$\sigma_{\text{CRLB}}^2 = \frac{1}{8\pi^2 T f_0 B \left(1 + \frac{B^2}{32\rho^2} \right)} \left[ \frac{1}{\rho^2 \left(1 + \frac{1}{\text{SNR}}\right)^2} - 1 \right]. \quad (1)$$

To derive this expression, the following assumptions were made: both the signals and the noise have flat bandlimited spectra, the signal and noise are uncorrelated, and both the signals are corrupted by noise of equal power. (1) above and (20) in [2] differ slightly because the bandwidth is the fractional bandwidth and the signal-to-noise ratio is the ratio of

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rms amplitudes in [2]. This expression, however, is of limited utility in this form, since the CRLB gives the lowest attainable variance at high SNR only. At lower SNR, the attainable variance is larger than the CRLB. However, the CRLB will also underestimate the minimum attainable variance when the correlation between the signals is poor, even when the SNRs are very high. Unfortunately, it is impossible to know when the transition will occur if signal decorrelation is present.

Friedel [1] has derived a relationship between the SNRs in a pair of signals and their correlation coefficient when the signals and the noises were uncorrelated with zero-means. We have independently derived a relationship between the peak of the crosscorrelation coefficient function and the SNR for the case when the signals have a delay between them. The derivation is shown in the Appendix.

A derivation for the CRLB for signals that are corrupted by both noise and decorrelation follows. Decorrelation will be treated as another noise source since it can be expressed as a noise term using the formulation in the appendix. Then, to verify the SNR-p relationship, the derived CRLB will be shown to be equivalent to (1).

The minimum possible variance of estimates at high SNR is given by the CRLB [8]. For TDE, the CRLB can be expressed in terms of the signal parameters as follows [9]:

\[ \sigma^2(D - \bar{D}) \geq \sigma^2_{CRLB} = \left[ 2T \int_0^{\infty} (2\pi f)^2 \frac{1}{1 - |\gamma_{12}(f)|^2} df \right]^{-1}, \] (2)

where \( T \) is the data length, \( D \) is the time delay, \( \bar{D} \) is the time delay estimate, and

\[ |\gamma_{12}(f)|^2 = \frac{G_{ss}(f)}{G_{ss}(f) + G_{nn}(f)} \] (3)

is the magnitude-squared coherence function. The indexed \( G(f) \) are the one-sided power spectra of the signal and noise (signal and noise are assumed real). (2) can be used with signals and noise of arbitrary spectral shape and gives the lower limit on the estimation variance for relatively high SNR. At low SNR, the CRLB underestimates the achievable variance; other expressions have been proposed in such cases [16]. However, for partially correlated signals, similar situations may arise when the signals are poorly correlated even though the SNR is relatively high.

For most spectra, the integral in (2) is difficult to evaluate analytically. In such cases, a reasonable approximation to the CRLB can be obtained using the equivalent noise bandwidth [2],[10],[17]. The spectra are assumed to be flat between

\[ f_{\text{max}} = f_0 + B/2 \text{ and } f_{\text{min}} = f_0 - B/2, \] (4)

where \( B \) is the bandwidth and \( f_0 \) is the center frequency. In this case, integrating (2), we obtain

\[ \sigma^2_{CRLB} = \left[ 4\pi^2 T \left( f_{\text{max}}^3 - f_{\text{min}}^3 \right) \right]^{-1} \left( \frac{1 - |\gamma_{12}|^2}{|\gamma_{12}|^2} \right), \] (5)

where the magnitude-squared coherence is a constant inside the specified frequency band, and zero outside. Substituting (4) into (5) and simplifying, we obtain

\[ \sigma^2_{CRLB} = \frac{1}{8\pi^2 T f_0^3 B \left( 1 + \frac{B^2}{12 f_0^2} \right) \left( 1 - \frac{|\gamma_{12}|^2}{|\gamma_{12}|^2} \right)} \] (6)

In biomedical ultrasound applications, the signals are frequently approximated using Gaussian spectra. In such cases, an equivalent noise bandwidth can be used [21].

In ultrasonic imaging systems, electronic noise degrades system performance. Even in the absence of electronic noise, unwanted differences between reference and delayed signals influence blood velocity, tissue motion, and elasticity imaging techniques [22]. For blood velocity and tissue motion estimation techniques, the decorrelation is due to the transverse movement of scatterers through the transducer beam. For elasticity imaging based on tissue compression, the main source of decorrelation is the compression itself. The pre-compression and post-compression signals are not delayed replica of each other, but, are distorted due to tissue compression. These differences typically are described in terms of the correlation coefficient function. In the appendix, we show that if a signal is corrupted by noise sources that are uncorrelated with each other and with the signal, the noise can be expressed as a source of decorrelation. Conversely, decorrelation can also be treated as a source of additive noise as long as this noise can be modeled as stationary and uncorrelated with the signal. The assumptions of stationarity and uncorrelatedness are satisfied in blood velocity and tissue motion estimation, since some old scatterers leave the sample volume and new ones come in. In elasticity imaging, when the decorrelation due to compression is written as an additive noise term, it will generally be signal-dependent. Only at small compressions and/or small observation window size, the assumptions are approximately valid and all the cross terms can be ignored (as is done in (A.6)). Thus, the following discussion (which is done for any general case where decorrelation is present) is valid for elasticity imaging only at small compressions. However, when the assumptions are valid, all the classical analyses that involve the SNR can be used in connection with decorrelation as well. It should be noted that the techniques for imaging of elasticity and strain typically involve small applied strains (≤ 1%) [4].

In the following signal model, we have both the random noise and decorrelation (expressed as a noise term),

\[ r_1(t) = s_1(t) + n_{11}(t) + n_{12}(t) \] (7)

\[ r_2(t) = s_2(t) + n_{21}(t) + n_{22}(t) \] (8)

where the random noise terms \( n_1(t) \) and \( n_2(t) \) have equal power and are uncorrelated with the signal and the decorrelation noise. We assume that the decorrelation noise is uncorrelated with the signal and the random noise.

For the above model, the magnitude-squared coherence can be expressed as

\[ |\gamma_{12}(f)|^2 = \frac{G_{ss}(f)}{G_{ss}(f) + G_{nn}(f)} \] (9)
This expression can be rewritten in terms of SNR, viz:

\[ |\gamma_{12}(f)|^2 = \frac{1}{\left(1 + \frac{1}{SNR_\rho} + \frac{1}{SNR}\right)^2} . \]  (10)

The assumption is that the signal, decorrelation noise, and random noise have flat bandlimited spectra. Substituting (10) in (6), we obtain

\[ \sigma^2_{CRLB} = \frac{1}{8\pi^2 T f_0^2 B \left(1 + \frac{\beta^2}{12 f_0^2}\right)} \left[\left(1 + \frac{1}{SNR_\rho} + \frac{1}{SNR}\right)^2 - 1\right] . \]  (11)

The above expression shows two SNR terms for the decorrelation noise and the electronic noise, respectively. When the decorrelation SNR, is much lower than the random SNR (e.g., a situation that arises in elasticity estimation when there is relatively large strain), we obtain

\[ \sigma^2_{CRLB} \approx \frac{1}{8\pi^2 T f_0^2 B \left(1 + \frac{\beta^2}{12 f_0^2}\right)} \left[\left(1 + \frac{1}{SNR_\rho}\right)^2 - 1\right] . \]  (12)

When both SNR and SNR, are large-valued (SNR, \(\gg 1\) and SNR \(\gg 1\)), we have

\[ \left[\left(1 + \frac{1}{SNR_\rho} + \frac{1}{SNR}\right)^2 - 1\right] \approx 2 \left[\frac{1}{SNR_\rho} + \frac{1}{SNR}\right] , \]  (13)

so,

\[ \sigma^2_{CRLB} \approx \frac{1}{4\pi^2 T f_0^2 B \left(1 + \frac{\beta^2}{12 f_0^2}\right)} \left[\frac{1}{SNR_\rho} + \frac{1}{SNR}\right] . \]  (14)

Using the SNR-\(\rho\) relationship (A.13), we will now demonstrate that (14) is equivalent to the CRLB expression in (1).

Following the notations in [2] (note the difference between this model and the one described by (A.1) and (A.2)),

\[ r_1(t) = s(t) + n_1(t) \quad \text{and} \quad r_2(t) = s_2(t) + n_2(t) , \]  (15)

where \(n_1(t)\) and \(n_2(t)\) are uncorrelated random noises, and \(s_1(t)\) and \(s_2(t)\) are decorrelated echo signals that would have been received in the absence of the random noise. Assuming that \(s_1(t)\) and \(s_2(t)\) have the same power spectra, the decorrelation between them can be expressed as noise terms. Thus,

\[ r_1(t) = s(t) + n_{12}(t) + n_1(t) \quad \text{and} \quad r_2(t) = s(t) + n_{22}(t) + n_2(t) , \]  (16)

Now, we show below that starting from (17) and (18), we can derive (1) that verifies the SNR-\(\rho\) relationship derived in the appendix. Let us assume that the SNRs in the signals shown in (15) and (16) are equal, and the random and decorrelation noise are uncorrelated with each other and with the signal. We also assume that the signal and the noise have flat power spectra over a limited frequency band. Then, rather than expressing the SNR as the ratio of area under the power spectra for signal and the noise, it can also be expressed as a ratio of signal and noise power spectra at any frequency within the passband. Note that in [2], SNR was defined as a ratio of RMS amplitudes, and thus was the square root of the SNR as defined in this paper. Then,

\[ SNR = \frac{G_{s_1,s_1}(f)}{G_{n_n}(f)} = \frac{G_{s_2,s_2}(f)}{G_{n_n}(f)} = \frac{G_{s_1}(f) + G_{n_{12},n_{12}}(f)}{G_{n_n}(f)} . \]  (19)

Here we have made use of (17) and (18) to expand the power spectral density (PSD) of the decorrelated signals into PSDs of the signal plus the equivalent decorrelation noise.

For this signal model, the magnitude-squared coherence is given by

\[ |\gamma_{12}(f)|^2 = \frac{G_{s_1,s_2}(f)}{G_{s_1}(f) + G_{n_{12},n_{12}}(f) + G_{n_n}(f)} \]  (20)

Multiplying and dividing the third term in the denominator by \([G_{s_1}(f) + G_{n_{12},n_{12}}(f)]\), we get

\[ |\gamma_{12}(f)|^2 = \frac{1}{\left[1 + \frac{G_{n_{12},n_{12}}(f)}{G_{s_1}(f)}\right] \left[1 + \frac{G_{n_n}(f)}{G_{s_1}(f)}\right]} \]  (21)

Using (19),

\[ |\gamma_{12}(f)|^2 = \frac{1}{\left[1 + \frac{G_{n_{12},n_{12}}(f)}{G_{s_1}(f)}\right] \left[1 + \frac{G_{n_n}(f)}{G_{s_1}(f)}\right]} \]  (22)

Using (A.12), we obtain \(\frac{1}{\rho} = 1 + \frac{1}{SNR_{\rho}}\), so that

\[ |\gamma_{12}(f)|^2 = \frac{1}{\rho^2 \left[1 + \frac{1}{SNR}\right]^2} \]  (23)

Thus,

\[ \frac{1 - |\gamma_{12}(f)|^2}{|\gamma_{12}(f)|^2} = \frac{1}{|\gamma_{12}(f)|^2} - 1 = \frac{1}{\rho^2 \left[1 + \frac{1}{SNR}\right]^2} - 1 , \]  (24)

and, substituting the above in (6), we obtain

\[ \sigma^2_{CRLB} = \frac{1}{8\pi^2 T f_0^2 B \left(1 + \frac{\beta^2}{12 f_0^2}\right)} \left[\frac{1}{\rho^2 \left[1 + \frac{1}{SNR}\right]^2} - 1\right] \]  (25)

which is identical to (1).
A. Verification of the SNR-\(\rho\) relationship

Using LabVIEW, we developed a program to test the validity of (A.12). We chose to set the delay between the signals, \(t_0 = 0\), without loss of generality. Three uncorrelated Gaussian white noise sequences were created to represent the reference signal and two noise sequences. The noise was scaled and added to the reference signal to produce SNRs ranging from -20 dB to 80 dB. To obtain better averaging of the results, the length of the sequences was 2000 points for SNR > 20 dB and 10000 points for SNR < 20 dB.

The crosscorrelation coefficients estimated from the data at the specified SNRs are plotted as a solid line in Fig. 1. Using (A.12), we computed the correlation coefficient from the known SNRs; these are shown by the broken line in Fig. 1. The results calculated from (A.12) lie on the solid line, demonstrating excellent agreement and the validity of the SNR-\(\rho\) expression.

III. SUMMARY AND CONCLUSION

When tissue under an ultrasonic transducer moves or is deformed, the scatterers move along and across the axis of the beams. In medical ultrasound, this relative motion between the beam and the tissue results in decorrelation that adds to other sources of random noise and limits the performance of displacement estimation algorithms.

Decorrelation is usually expressed in terms of correlation coefficient functions. Friemel [1] derived a relationship between SNR and the correlation coefficient when a zero-mean signal is corrupted by uncorrelated zero-mean noise. In this paper, an expression is derived that converts the peak of the correlation coefficient function into an SNR. This expression is valid when the noise is stationary and uncorrelated with the signal. The validity of the expression is corroborated using a simulation. In this simulation, we assume that there is no delay between the two signals (\(t_0 = 0\)) without loss of generality, and we use the correlation coefficient in place of the peak of the crosscorrelation coefficient function. The SNR and the correlation coefficient are nonlinearly related; Fig. 1 illustrates that, whereas for SNRs above 20 dB the correlation remains high (\(\rho > 0.98\)), the correlation decreases abruptly for SNRs below 20 dB. Thus, it is of practical interest to note that below \(\rho = 0.98\), a small decrease in correlation coefficient means a large decrease in the corresponding SNR.

The SNR-\(\rho\) relationship can be very useful in evaluating various TDE algorithms when applied on partially correlated data. An expression was derived in [2] where the CRLB is given in terms of \(\rho\) and the SNR. The utility of that expression is uncertain since the Cramér-Rao bound itself is customarily described only in terms of the SNR; one may be operating altogether outside the CRLB due to a low value of \(\rho\), yet not be cognizant of this fact based on [2]. If the correlation between the signals is also expressed in terms of an SNR, then the CRLB and the thresholds of applicability of CRLB can be expressed in terms of a unified SNR and would be easier to apply. We have derived an expression for CRLB based on the equivalence of SNR and peak of the correlation coefficient function. We also demonstrated using the SNR-\(\rho\) relationship that the CRLB expression based on the equivalent SNR is identical to the expression given in [2].

This SNR-\(\rho\) relationship is of interest in the work on blood velocity, tissue motion estimation, and elasticity imaging, to name a few. However, this relationship can be used for converting the decorrelation due to compression to SNR in elasticity imaging for small strains only. For large strains, the assumption of stationarity and uncorrelatedness no longer remains valid; thus, care should be taken when using this expression for conversion. This SNR-\(\rho\) relationship can be used to convert correlation coefficient values to SNR to compare the performances of elastography under different compressions, as long as these assumptions remain valid. An example is the derivation of the strain filter for elastography [23].

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APPENDIX

THE RELATIONSHIP BETWEEN THE CORRELATION COEFFICIENT FUNCTION AND THE SNR:

We model two signals \(x(t)\) and \(y(t)\), where \(x(t)\) is the signal \(s(t)\) corrupted by noise \(n_1(t)\), and \(y(t)\) is a delayed version of \(s(t)\) corrupted by noise \(n_2(t)\). The noise terms \(n_1(t)\) and \(n_2(t)\) are uncorrelated and stationary.

\[
x(t) = s(t) + n_1(t) \quad \text{and} \quad \tag{A.1}
\]
\[
y(t) = s(t - t_0) + n_2(t) \quad . \tag{A.2}
\]

The value of the crosscorrelation coefficient function between \(x(t)\) and \(y(t)\) at the delay time \(t_0\) is given in terms of correlation functions by [20, p. 536]:

\[
\rho_{xy}(t_0) = \frac{R_{xy}(t_0)}{\sqrt{R_{xx}(0)R_{yy}(0)}} \quad . \tag{A.3}
\]

Using the Wiener-Khinchine relations the correlation functions can be expressed as inverse Fourier transforms of power spectral
we obtain

\[ R_{xy}(t_0) = \int_{-\infty}^{\infty} G_{xy}(f)e^{-j2\pi ft_0}df, \]
\[ R_{xx}(0) = \int_{-\infty}^{\infty} G_{xx}(f)df, \quad \text{and} \]
\[ R_{yy}(0) = \int_{-\infty}^{\infty} G_{yy}(f)df. \]  

(A.4)

Given that the signal and noise are uncorrelated, the power autospectra and cross-spectra are given by [20, p. 177]:

\[ G_{xy}(f) = G_{ss}(f)e^{-j2\pi ft_0} + G_{s1}(f) + G_{s2}(f) + G_{n1}(f)G_{n2}(f) = G_{ss}(f)e^{-j2\pi ft_0}, \]
\[ G_{xx}(f) = G_{ss}(f) + G_{n1}(f), \]
\[ G_{yy}(f) = G_{ss}(f) + G_{n2}(f), \]
\[ G_{nn}(f) = G_{n1}(f)G_{n2}(f) = G_{nn}(f). \]  

(A.6)  

(A.7)  

(A.8)  

(A.9)

Then, substituting in (A.5) yields

\[ R_{xy}(t_0) = \int_{-\infty}^{\infty} G_{xy}(f)e^{-j2\pi ft_0}df \]
\[ = \int_{-\infty}^{\infty} G_{ss}(f)e^{-j2\pi ft_0}e^{-j2\pi ft_0}df \]
\[ = \int_{-\infty}^{\infty} G_{ss}(f)df \]
\[ R_{xx}(0) = R_{yy}(0) = \int_{-\infty}^{\infty} G_{xx}(f)df \]
\[ = \int_{-\infty}^{\infty} \left\{ G_{ss}(f) + G_{nn}(f) \right\}df. \]  

(A.10)

The signal-to-noise ratio (the ratio of signal power to noise power in either of the two corrupted signals, assumed to be the same in both) can be expressed as

\[ SNR = \frac{\int_{-\infty}^{\infty} G_{ss}(f)df}{\int_{-\infty}^{\infty} G_{nn}(f)df}. \]

(A.11)

Substituting (A.10) into (A.3), we obtain

\[ \rho_{xy}(t_0) = \frac{\int_{-\infty}^{\infty} G_{ss}(f)df}{\int_{-\infty}^{\infty} G_{ss}(f) + G_{nn}(f)df} \]
\[ = \frac{\int_{-\infty}^{\infty} G_{ss}(f)df}{\int_{-\infty}^{\infty} G_{nn}(f)df + 1} = \frac{SNR}{SNR + 1} \]
\[ = \frac{\rho_{xy}(t_0)}{1 - \rho_{xy}(t_0)}. \]  

(A.12)  

(A.13)

Thus, if a signal and its delayed version are corrupted by uncorrelated noise renditions \( n_1(t) \) and \( n_2(t) \), such that \( SNR_1 = SNR_2 = SNR \leq \infty \), the correlation coefficient function at delay \( t_0 \), \( \rho_{xy}(t_0) \) is uniquely related to SNR. Note that \( \rho_{xy}(t) \) is expected to maximize at \( t_0 \). Thus, the corruption of signals with noise can be characterized by either the SNR or a decorrelation measure, namely, the peak value of the cross-correlation coefficient function. When both decorrelation and random noise are present, the characterization can be done either with an SNR and a decorrelation measure or, preferably, with an equivalent SNR measure.

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